8
Applying Vibration Models

As we noted in Chapter 7, vibration is omnipresent in our lives, both in people-made and living objects and devices. Vibration is also complex. For example, sound is modeled as a sum of *harmonics*, of vibrations with different periods or natural frequencies. Certainly buildings and cars and airplanes and dentists' drills vibrate in complex, *multi-modal* ways as well, with a lot of modes having different frequencies and different amplitudes. Given that life seems so complex, is it worth doing elementary vibration modeling? Yes, it is, as so eloquently said by one of the great pioneers of the field of vibration, Sir John William Strutt, third Baron Rayleigh, known quite widely as Lord Rayleigh:

The material systems, with whose vibrations Acoustics is concerned, are usually of considerable complication, and are susceptible of very various modes of vibration, any or all of which may coexist at any particular moment. Indeed in some of the most important musical instruments, as strings and organ-pipes, the number of independent modes is theoretically infinite, and the consideration of several of them is essential to the most practical questions relating to the nature of the consonant chords. Cases, however, often present themselves, in which one mode is of paramount importance; and even if this were not so, it would still be proper to commence the consideration of the general problem with the simplest case—that of one degree of freedom. It need not be supposed that the mode treated of is the only one possible, because so long as vibrations of other modes do not occur their possibility under other circumstances is of no moment.

Guided by Lord Rayleigh's insight, we will continue to limit our discussion of models of vibratory behavior to those having but a single degree of **Why?**

freedom. We will focus on two important elements. First, we develop the *mechanical-electrical analogy*, wherein we make more explicit the several commonalities of vibration behavior that we had identified in Chapter 7. In our second focus, we note a dividing line that is extraordinarily powerful for modeling vibration: some phenomena seem to go on indefinitely, quite on their own, while others appear as responses to repetitive stimulation. Thus far, our models have been in the first category, called *free* or *unforced* vibration, referring to phenomena that continue after some initial jolt gets them going. It includes the vibration of struck piano strings and the tides of the seas. The second category that we take up in this chapter, *forced* vibration, occurs when there is a persistent, repetitive input, such as the kind an air conditioning system imparts to the building it cools or an engine imparts to the vehicle it powers.

8.1 The Spring–Mass Oscillator–II: Extensions and Analogies

How? In Section 7.3 we noted that the pendulum could be modeled as a *spring-mass oscillator*, a model we now develop by applying once again the force balance embodied in Newton's second law. We show such a spring-mass system in Figure 8.1. Newton's law states that (see Section 7.3.1) the motion of the oscillator's mass, m, is governed by

$$\text{net force} = m\frac{d^2x(t)}{dt^2}. \tag{8.1}$$

Given? Two forces are shown acting on the mass: a specified *applied force*, $F(t)$, and a force exerted by the spring. The spring is an ideal elastic spring that has no mass and dissipates no energy. Its attachment points at each end

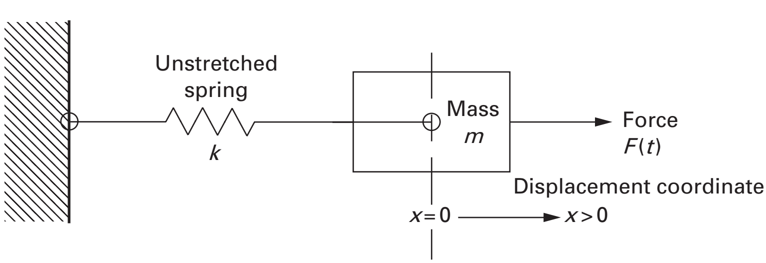


Figure 8.1 An elementary *spring-mass system* the shows an ideal spring exerting a restoring force on a mass, m, as does a specified applied force, $F(t)$. The spring's stiffness is k, and the displacement or movement of the mass to which the spring's right end is attached is $x(t)$.

are called *nodes.* The left node of the spring in Figure 8.1 is attached to a fixed point, say on a wall, while the right node is attached to a mass whose movement, $x(t)$, is the system's single degree of freedom. Moreover, the spring always exerts a *restoring force* on the node or mass that returns the spring to its original, unextended position. Thus, if moved a positive distance to the right, $x(t)$, the spring pulls the node back to the left; if the spring is compressed a distance to the left, $-x(t)$, it pushes the node back to the right. The magnitude of the spring force is given by **Assume?**

$$F_{\text{spring}} = kx(t). \tag{8.2}$$

The net force on the mass is the difference between the applied and the spring forces,

$$\text{net force} = F(t) - F_{\text{spring}}. \tag{8.3}$$

so that the equation of motion is found by combining eqs. (8.1), (8.2), and (8.3):

$$m\frac{d^2x(t)}{dt^2} + kx(t) = F(t). \tag{8.4}$$

Equation (8.4) was already introduced as an analog of the pendulum in Section 7.3, where we made the argument that the gravitational pull on the pendulum mass exerted a spring-like force on the pendulum (see Problem 8.1). For free, unforced vibration, there is no applied force, and the governing equation is

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0. \tag{8.5}$$

If we introduce a scaling factor, ω_0, to make the time dimensionless, as we did in eq. (7.10), the oscillator equation (8.5) becomes

$$m\omega_0^2\frac{d^2x(\tau)}{d\tau^2} + kx(\tau) = 0, \tag{8.6}$$

which suggests that the scaling factor for the spring-mass system is

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{8.7}$$

Equation (8.7) can be confirmed to be dimensionally correct (see Problem 8.2) and, as for the pendulum, ω_0 can be identified as the *circular frequency* of the spring-mass oscillator. The circular frequency can be related to the frequency and the period:

$$f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{8.8}$$

Again, both f_0 and ω_0 have the physical dimensions of $(\text{time})^{-1}$, but the units of f_0 are number of cycles per unit time, while those of ω_0 are radians per unit time.

Use? Predict? Equation (8.7) is actually far more important than its simple appearance suggests. It provides a fundamental paradigm for thinking about the vibration of systems: The natural frequency of the oscillator is proportional to the square root of the *stiffness-to-mass* ratio. Thus, natural frequencies increase (and periods decrease) with increasing stiffness, k, while natural frequencies decrease (and periods increase) with increasing mass, m. We will refer back to this paradigm often, and we will also see that it captures a very useful design objective.

Why? How? We now extend the spring-mass model to incorporate non-ideal, dissipative behavior. We do this by attaching to the mass a *damping* or *dissipative element*, sometimes called a *dashpot* or *damper*, which exerts a restoring force proportional to the speed at which the element is extended or compressed:

$$F_{\text{damper}} = c\dot{x}(t). \tag{8.9}$$

The damper acts *in parallel* with the spring, as shown in Figure 8.2, so that the net force exerted on the mass is

$$\text{net force} = F(t) - F_{\text{spring}} - F_{\text{damper}}, \tag{8.10}$$

and the corresponding equation of motion for a *spring-mass-damper system* is

$$m\frac{d^2x(t)}{dt^2} + c\dot{x}(t) + kx(t) = F(t). \tag{8.11}$$

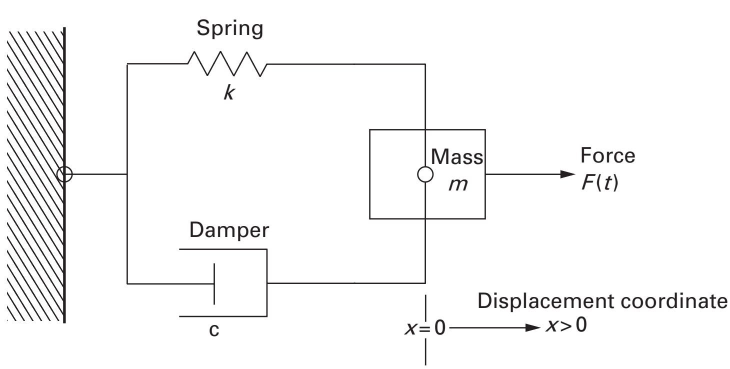


Figure 8.2 An elementary *spring-mass-damper system* that shows the ideal spring (of stiffness, k) exerting a force on a mass, m, the specified applied force, $F(t)$, and a viscous damping element that exerts a restoring force that is proportional to the speed, $\dot{x}(t)$, at which the mass moves.

This result is very similar to the corresponding result for the damped pendulum, eq. (7.27), save for the facts that the present result includes a forcing function, $F(t)$, and its spring term is (already) linear.

Problem 8.1. We experience the pull of gravity as constant and not dependent on position. How does it come to be interpreted as exerting a spring force that is linearly proportional to position? (*Hint*: Think about the equation of motion in which the relevant term appears.)

Problem 8.2. Identify the fundamental physical dimensions of the spring stiffness, k, and the mass, m, and use them to determine the physical dimensions of ω_0 for a spring-mass oscillator.

8.1.1 Restoring and Dissipative Forces and Elements

Equation (8.11) offers the prospect of generalizing the energy ideas of Sections 7.1.5 and 7.1.6 in rather broad terms. The spring-mass-damper system is itself a paradigm for a very broad range of vibration models—physical, biological, chemical, and so on. Thus, we will not only be able to identify a system's mass, but we will also be able to identify a spring-like element with a stiffness, such as the gravitational pull of the pendulum, and a dissipative element with a damping constant, much like the shock absorber of an auto suspension (see Section 8.3). There is one salient feature common to each of these elements that will be true no matter what physical, biological, chemical or other model we are analyzing: Each element either *stores* energy or *dissipates* energy. Two elements store energy in the spring-mass-damper: the mass, which stores kinetic energy,

$$KE = \frac{1}{2}m(\dot{x}(t))^2, \tag{8.12}$$

and the spring, which stores potential energy,

$$PE = \frac{1}{2}k(x(t))^2. \tag{8.13}$$

In an ideal system, where there is no damping, the spring and the mass exchange energy from potential to kinetic to potential, and so on indefinitely. Thus, the two storage elements exchange their forms of energy repetitively as the ideal spring-mass system vibrates.

The damping element dissipates energy according to (see eq. (7.29))

$$\frac{dE(t)}{dt} = -\frac{1}{2}c(\dot{x}(t))^2. \tag{8.14}$$

As a spring-mass-damper vibrates or oscillates, energy is no longer simply passed back and forth between the spring and the mass. Rather, the damping element draws energy out of the system and dissipates it as wasted power or energy, typically through the heat transfer we associate with frictional devices.

Again, these characterizations turn out to be useful for helping us analyze systems or phenomena as we try to build models of their behavior.

8.1.2 Electric Circuits and the Electrical-Mechanical Analogy

Electric circuits and their elements offer a parallel paradigm for analyzing oscillatory behavior. Consider the elementary, *parallel RLC circuit* shown in Figure 8.3. It has three ideal elements connected in parallel that are driven by a *current source* that produces a current $i_{\text{source}}(t)$. The three elements are idealized in the same way that the mass of a spring-mass system is perfectly rigid and that its spring is mass-less. The first element we introduce is the ideal *capacitor* that, when discharged, transmits a voltage drop, $V(t)$, that is proportional to the electric charge, $q(t)$, stored on two plates separated

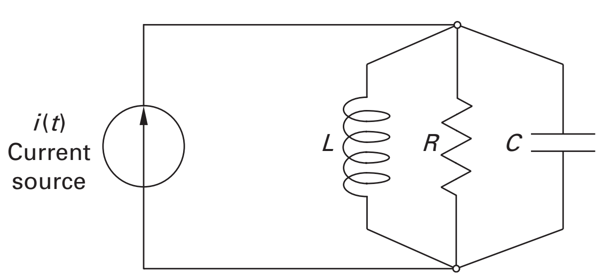


Figure 8.3 A *parallel RLC circuit* that has a current source as its driver. The elements are the *capacitor* of capacitance, C, the *inductor* with inductance, L, and the *resistor* with resistance, R. The current source provides a current of magnitude, $i_{\text{source}}(t)$.

by an insulator:

$$V(t) = \frac{q(t)}{C}. \tag{8.15}$$

The constant, C, is the *capacitance* of the capacitor and its units are farads, named after the British chemist and physicist Michael Faraday (1791–1867). The capacitor stores energy in an amount proportional to the square of the voltage across it:

$$E_C = \frac{1}{2}C\,(V(t))^2\,. \tag{8.16}$$

Notwithstanding the elegant simplicity of eqs. (8.15) and (8.16), electrical circuit models are generally cast in terms of the time rate of change of charge, called the *current*, because it is hard to measure charge:

$$i(t) = \frac{dq(t)}{dt}. \tag{8.17}$$

This form of the capacitor model is an element that carries a current, $i_C(t)$, that is directly proportional to the time rate of change of the voltage drop, $V(t)$, across the capacitor:

$$i_C = C\frac{dV(t)}{dt}. \tag{8.18}$$

The second element we introduce is the *inductor*, which is a coil that builds up a magnetic field rate when a current flows through it. The magnetic field causes a voltage drop across the inductor that is proportional to the time rate of change of the current flowing through it:

$$\frac{di_L}{dt} = \frac{V(t)}{L}. \tag{8.19}$$

The constant, L, is the *inductance*, which is measured in henrys, named after the American physicist Joseph Henry (1797–1878). Now we integrate eq. (8.19) with respect to time,

$$i_L = \frac{1}{L}\int_{-\infty}^{t} V(t')dt', \tag{8.20}$$

where t' is a dummy variable of integration in the integral in eq. (8.20). The inductor stores energy in an amount proportional to the square of the current flowing through it:

$$E_L = \frac{1}{2}L(i_L(t))^2 = \frac{1}{2L}\left(\int_{-\infty}^{t} V(t')dt'\right)^2. \tag{8.21}$$

The third element is the *resistor*. It impedes (or resists) the flow of charge in proportion to the time rate of change of charge, or the current. The resulting voltage drop across the resistor is directly proportional to the current flowing through it:

$$i_R = \frac{V(t)}{R}, \tag{8.22}$$

where the constant, R, is the *resistance*, which is measured in ohms, named after the German physicist Georg Simon Ohm (1787–1854). The resistor, like its mechanical counterpart, the dashpot, dissipates energy by throwing it off as waste heat or power. Thus, in the context of Section 8.1.1, we can regard the resistor and the dashpot as similar dissipative elements, and the capacitor (like the mass) and the inductor (like the spring) as elements that store energy.

Why? Can we draw an analogy between the electrical elements just introduced and the spring-mass-damper system described earlier in this section? Yes. In fact, there are two well-known electrical-mechanical analogies. The choice of analogy is to some extent a matter of taste, and we describe here the one we prefer; this book's first edition presented the other.

How? We first invoke Gustav Robert *Kirchhoff's* (1824–1887) *current law* (KCL) to derive the governing equations for the parallel *RLC* circuit in Figure 8.3. The KCL states that the time rate of change of the electrical charge flowing into or out of a node or connection in a circuit must be zero. In other words, a node cannot accumulate charge. Expressed mathematically, the KCL states that

$$\frac{dq_{\text{node}}(t)}{dt} = \sum_{n=1}^{N} i_n(t) = 0, \tag{8.23}$$

where the $i_n(t)$ are the currents taken as positive flowing into the node through the N elements connected at that node. Thus, looking at the indicated currents going into and out of either of the two nodes in Figure 8.3, we see that

$$\sum_{n=1}^{N} i_n(t) = i_{\text{source}}(t) - i_C - i_L - i_R = 0, \tag{8.24}$$

where, again, $i_{\text{source}}(t)$ is the current provided by the *current source* in the circuit, and the remaining terms are the currents flowing through the capacitor, the inductor, and the resistor, respectively. Note that eq. (8.24) looks remarkably like a force balance equation [e.g., eqs. (8.3) and (8.10)]! We now replace the currents in the elements by their respective *constitutive equations* (8.18), (8.20), and (8.22), that describe how the current flows through each relates to the voltage across each. Then eq. (8.24) becomes:

$$C\frac{dV(t)}{dt} + \frac{V(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} V(t')dt' = i_{\text{source}}(t). \tag{8.25}$$

If we differentiate eq. (8.25) once with respect to time, we find:

$$C\frac{d^2V(t)}{dt^2} + \frac{1}{R}\frac{dV(t)}{dt} + \frac{1}{L}V(t) = \frac{di_{\text{source}}(t)}{dt}. \tag{8.26}$$

Equation (8.26) is a second-order, linear differential equation with constant coefficients. Its dimensions can be shown to be consistent and correct (see Problem 8.4). When solved, it yields the common voltage across the three parallel elements, from which both the currents through each and the energy stored by the capacitor and inductor can be calculated [using eqs. (8.18), (8.20), and (8.22)]. **Use? Predict?**

What is most noteworthy about eq. (8.26) is its uncanny resemblance to eq. (8.11), the equilibrium equation for the spring-mass-damper. It is most tempting to conclude that voltage is analogous to displacement, and that

$$C \sim m, \quad \frac{1}{R} \sim c, \quad \frac{1}{L} \sim k. \tag{8.27}$$

Some further expressions of this *electrical-mechanical analogy* are shown in Table 8.1. The analogy is interesting and useful. Consider, for example, the fact that we described the *RLC* circuit in Figure 8.3 as a parallel circuit. In the spring-mass-damper of Figure 8.2, we specifically inserted the dashpot as an element in parallel with the spring. The mass can also be said to be in parallel with the spring and the dashpot since it shares their common endpoint displacement. Further, the analogy extends into the context of system characterization: A system can be said to be very stiff if k is large or its inductance, L, is small, or as having a large effective mass or inertia if either its mass, m, or its capacitance, C, is large.

Now, to complete this introduction to the electrical-mechanical analogy, we repeat the thought that the choice of analogies is a matter of taste. The analogy presented here allows us to draw distinctions between behaviors that go *through* elements (force and current), and those measured *across* elements (displacement and voltage). The analogy also enables us to identify Newton's second law and Kirchhoff's current law as similar expressions of balance (force or current) or conservation (momentum or charge). The other analogy identifies force with voltage and displacement with charge. It, therefore, does offer some more immediately recognizable appeal because the resemblance of basic equations is even more obvious.

Table 8.1 Elements of one electrical-mechanical analogy.

Mechanical	Electrical
Momentum ($\sim$ Speed): $mv(t)$	Charge: $q(t)$
Force ($\sim d$(Momentum)$/dt$): $F = m\dfrac{dv(t)}{dt}$	Current($\sim d$: (Charge)$/dt$): $i(t) = \dfrac{dq(t)}{dt}$
Displacement: $x(t)$	Voltage: $V(t)$
Newton's 2nd @Massless Node: $\sum_{n=1}^{N} F_n(t) = \dfrac{d(mv_{\text{node}}(t))}{dt} = 0$	Kirchhoff's Current Law: $\dfrac{dq_{\text{node}}(t)}{dt} = \sum_{n=1}^{N} i_n(t) = 0$
$F_{\text{spring}} = k \int_{-\infty}^{t} v(t')dt' = kx(t)$	$i_L = \frac{1}{L} \int_{-\infty}^{t} V(t')dt'$
$F_{\text{damper}} = cv(t) = c\dot{x}(t)$	$i_R = \frac{1}{R} V(t)$
$F_{\text{net}} = m\dot{v}(t) = m\dot{x}(t)$	$i_C = C\dot{V}(t)$
$PE = \frac{1}{2}k\,(x(t))^2$	$E_C = \frac{1}{2}C\,(V(t))^2$
$KE = \frac{1}{2}m\,(\dot{x}(t))^2$	$E_L = \frac{1}{2}L\,(i(t))^2 = \frac{1}{2}L\left(\dot{q}(t)\right)^2$

However, the preferred analogy described above is more consistent with physical principles and conforms better to our intuition of how such systems behave.

Problem 8.3. Taking as fundamental the dimensions of current, I, as charge per unit time and voltage (or *electromotive force*), V, as (force $\times$ distance) per unit charge, determine the fundamental physical dimensions of the capacitance, C, the inductance, L and the resistance, R.

Problem 8.4. Using the fundamental dimensions identified in Problem 8.3, confirm that eq. (8.26) is dimensionally consistent and correct.

Problem 8.5. Using the fundamental dimensions identified in Problem 8.3, determine whether the energy expressions for E_C and E_L given in Table 8.1 are dimensionally correct.

Problem 8.6. Determine the governing equation for the free oscillation of the voltage in a parallel LC circuit with ideal elements.

Problem 8.7. Determine the natural frequency of free vibration and the period of the ideal parallel LC circuit of Problem 8.6.

8.2 The Fundamental Period of a Tall, Slender Building

It is not surprising that buildings, especially tall and slender buildings, respond to several kinds of forces by vibrating. Buildings respond to aerodynamic forces set in motion by wind or by aircraft passing nearby. They also respond to ground-borne motion induced by traffic, earthquakes, or even explosions. These various inputs force not only the vibration of the building as a whole, but also its internal components (such as walls, floors, and windows). Further, most tall buildings have their own internal sources of vibration; for example, air conditioning systems, escalators, and elevators. What is most noteworthy is that tall buildings tend to be built lighter and with more flexibility than were earlier tall buildings (see Figure 8.4). For example, the Empire State Building is a good bit heavier and stiffer than the Sears Tower in Chicago (or were the towers of the World Trade Center in New York). As a result, building vibration, both local to a room and global to the building, has become a critical element in building design: vibration can create problems of annoyance, dysfunction, and outright danger for a building's occupants. The assessment of the

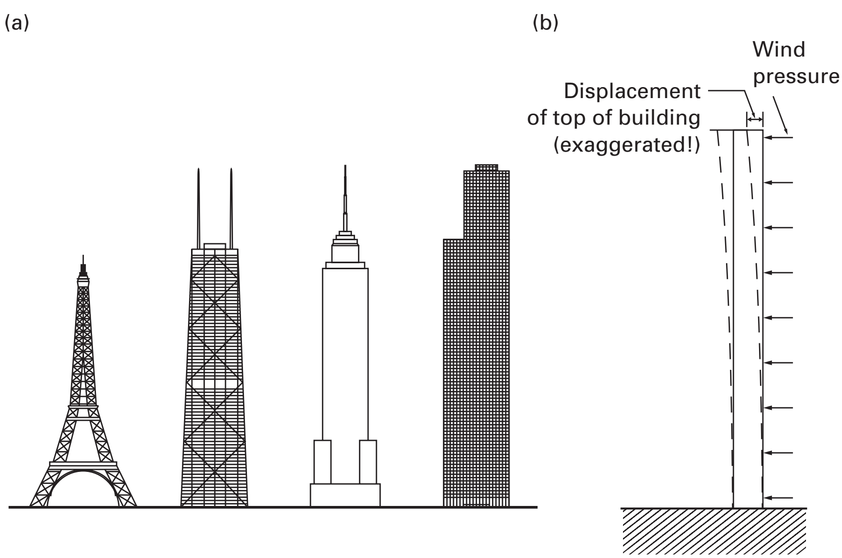


Figure 8.4 A small collection of skyscrapers, including the Eiffel Tower, the John Hancock center, the Empire State Building and the Sears Tower (after Billington, 1983). They are (mostly) tall and slender buildings that grace the skylines of modern cities. Also shown is a generic schematic of the greatly exaggerated movement of such a tall building in response to wind loading.

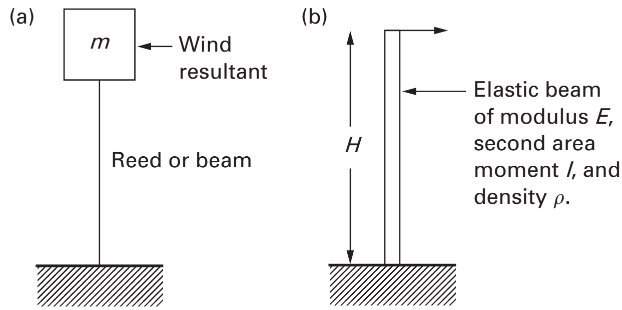


Figure 8.5 Two simple models used to estimate the fundamental period of vibration of tall, slender buildings: (a) a simple spring-mass system that is shown as a reed with a mass at its end; and (b) a cantilever beam model wherein the stiffness and the mass are distributed (uniformly) over the building height, H, but from which a simple spring-mass system can be deduced.

vibration response of a tall building, or any such structure, requires deep understanding of the building's dynamic properties, such as its own fundamental period or its natural frequency. It turns out, interestingly enough, that a "first-order" estimate of the natural frequency or period of a tall building can be obtained by making a lot of assumptions and modeling the entire building as a simple spring-mass system.

Consider the generic skyscraper shown in Figure 8.4, together with profiles of some real counterparts. We assume that the wind pressure is uniform over the building height and oriented normal to the side shown. The wind pressure produces a net force that pushes on that building face, thus making the building bend, with the largest movement at its free end at the top. Since buildings are made up of elastic structural members, which are themselves springs, we expect that the building will resist the bending motion caused by the wind and return to its original straight configuration when the wind ceases. In this sense, we can draw the building as a whole as if it were a simple elastic reed with a mass concentrated at its free end [see Figure 8.5(a)], but where this reed-and-mass system is exactly the same as the spring-mass system defined in Sections 7.3 and 8.1. We need only determine the stiffness, k, and the mass, m.

Assume?

One way to determine the stiffness of a building is to measure its deflection while a load or force is being applied to the building and then back-calculate the stiffness. (For a yet-to-be-built design, a similar measurement could be made on a comparable building.) Consider, for

Given?

example,a recently-built building with a *square* cross-section, $B = 30\,\text{m}$ (98.4 ft), on a side, and of height, $H = 300\,\text{m}$ (984 ft). (For a working calculation in standard American units, an experienced engineer would be likely to use $B = 100$ ft and $H = 1000$ ft.) A very strong, gale-force wind, say 100 mph (44.7 m/sec), produces a pressure of $1.23\,\text{kN/m}^2$ ($25.7\,\text{lb/ft}^2$) on the building, or a total wind force of

$$\text{wind force} = \begin{cases} 1.23\,\text{kN/m}^2 \times 30\,\text{m} \times 300\,\text{m} \\ 25.7\,\text{lb/ft}^2 \times 98.4\,\text{ft} \times 984\,\text{ft} \end{cases} = \begin{cases} 11.1 \times 10^6\,\text{N} \\ 2.49 \times 10^6\,\text{lb} \end{cases} \tag{8.28}$$

We will assume that the resultant of this force acts halfway up the building. **Assume?** The building will bend or move when it is loaded. A practical estimate is **Given?** that the top of the building will move about 0.3% of its height, or $0.003H$. Further, the deflection or movement of the building varies nonlinearly with **Assume?** height, so we will assume that the movement at that height is one-third of the movement at its top. With the building top expected to move 0.9 m (2.95 ft), we can calculate its stiffness as

$$k = \begin{cases} 11.1 \times 10^6\,\text{N} \div 0.30\,\text{m} \\ 2.49 \times 10^6\,\text{lb} \div 0.98\,\text{ft} \end{cases} = \begin{cases} 37.0 \times 10^6\,\text{N/m} \\ 2.54 \times 10^6\,\text{lb/ft} \end{cases} \tag{8.29}$$

To determine the building's fundamental period or natural frequency, we need its mass. A practical estimate of the weight of a building uses an average **Given?** specific weight of $\gamma = 1.50\,\text{kN/m}^3$ ($9.54\,\text{lb/ft}^3$) for a modern steel-framed tower with a 12 ft story height. In this case, the mass of the building can be calculated as:

$$m = \frac{\gamma H B^2}{g} = \begin{cases} [1.50\,\text{kN/m}^3 \times 300\,\text{m} \times (30\,\text{m})^2] \div 9.80\,\text{m/(sec)}^2 \\ [9.54\,\text{lb/ft}^3 \times 984\,\text{ft} \times (98.4\,\text{ft})^2] \div 32.2\,\text{ft/(sec)}^2 \end{cases}$$

$$= \begin{cases} 4.13 \times 10^7\,\text{kg} \\ 2.82 \times 10^6\,\text{lbm} \end{cases} \tag{8.30}$$

Thus, the fundamental period of this hypothetical generic skyscraper is

$$T_0 = 2\pi\sqrt{\frac{m}{k}} = 2\pi \begin{cases} \sqrt{4.13 \times 10^7\,\text{kg} \div 37.0 \times 10^6\,\text{N/m}} \\ \sqrt{2.82 \times 10^6\,\text{lbm} \div 2.54 \times 10^6\,\text{lb/ft}} \end{cases}$$

$$\cong \begin{cases} 6.64\,\text{sec} \\ 6.62\,\text{sec} \end{cases} \cong 6.6\ \text{sec}\,. \tag{8.31}$$

The result of eq. (8.31) is well within the range that experience suggests **Verified?** for the period of a modern, steel-framed building,which is about 5–10 sec for buildings whose height is within the range of 214–427 m (700–1400 ft). Another estimate is that the period of a building is within the range of

0.05–0.15 times the number of stories or floors. Since our hypothetical building is likely to have something like 85 floors, our estimate of its period is once again verified.

How? Another way to estimate the period or natural frequency of a building is to model it as a simple cantilever beam where the stiffness and mass are Given? distributed over the length of the beam [see Figure 8.5(b)]. The theory of strength of materials says that the stiffness of a cantilever beam of length, H, measured at the top, is given by

$$k_{\text{beam}} = \frac{3EI}{H^3}, \tag{8.32}$$

and that its period of vibration is given by

$$T_{\text{beam}} \cong 1.78H^2\sqrt{\frac{\gamma A}{gEI}}. \tag{8.33}$$

Here γ is, again, the specific weight of the beam (or building), A is the beam's cross-sectional area, I its *second moment of the cross-sectional area*, and E the *modulus of elasticity* of the material of which the beam is made. Given? Given that the dimensions of E are force per unit area and of I are (length)^4, it is easily verified that eq. (8.33) is dimensionally correct (see Problems 8.8–Predict? 8.9). Note that the stiffness decreases with H^3, while the period increases Use? with H^2, which means that its natural frequency also drops as H^2. Thus, a short building is stiffer than a tall building. In fact, the stiffnesses of two buildings made of the same material and having the same floor plan are related to each other as the cube of the inverse ratio of their heights.

It is also clear from eqs. (8.32) and (8.33) that the beam's stiffness increases with the product EI, and the period decreases with $1/\sqrt{EI}$. What do E, I, and their product EI mean for a beam and for a building? The modulus, E, represents the stiffness of the material of which the beam is made, and, not surprisingly, $E_{\text{steel}} > E_{\text{concrete}} > E_{\text{wood}}$. So, in very loose terms, a higher modulus is more suitable for taller buildings because of their higher material stiffness. (There are other issues involved, for example, the specific weight and the failure strength of materials, but that is well beyond our current modeling scope.) The second moment of the area, I, also (erroneously) called the *moment of inertia*, reflects the distribution of the cross-sectional area about its own centerline. For a building of square cross-section, $I \sim B^4$ roughly speaking, so that both the second moment of a building and its stiffness increase with its basic plan dimension to the fourth power, B^4.

This very brief overview of building vibrations suggests why engineers have had to worry only relatively recently about the effects of wind on tall buildings. Certainly tall structures were built long ago; one can point to the amazing cathedrals built during the Middle Ages (recall the discussion in Section 3.2.3), and even to the Eiffel Tower built in 1889. However, with the advent of both high-strength steels developed in the 20th century and new architectural styles, the flexible skyscraper came into being, bringing along both interesting problems and equally interesting opportunities. Thus, designing a building now means designing its dynamic properties and vibration response for sources of dynamic loading, including wind, earthquakes, nearby traffic, and mechanical systems within. Back-of-the-envelope estimates such as we have made play an important role in these designs because they enable engineers to make reasonable estimates of their designs long before they have to specify those designs to costly, detailed levels (see also Problems 8.41 and 8.42).

Problem 8.8. Given that the dimensions of the modulus of elasticity, E, are force per unit area, what are the dimensions of the second moment, I, that make eq. (8.32) dimensionally correct?

Problem 8.9. Using the dimensions identified in Problem 8.8, confirm that eq. (8.33) is dimensionally consistent and correct.

Problem 8.10. What is the pressure produced by a 100 mph wind expressed as a fraction of atmospheric pressure?

Problem 8.11. Show that the ordering of elastic moduli $E_{steel} > E_{concrete} > E_{wood}$ is correct in both metric and standard American units. (*Hint*: Use the library!)

Problem 8.12. For a tall cantilever of specific weight, γ, what are the physical dimensions of the parameter, $c \equiv \sqrt{E/(\gamma/g)}$? What could this parameter signify?

Problem 8.13. For the tall cantilever of Problem 8.12, with $I \sim B^4$, show that $T_0 \sim (H/c)(H/B)$. Is this result dimensionally meaningful?

8.3 The Cyclotron Frequency

To show that fundamental periods and frequencies are also important in other domains, we now present a simple model of the *cyclotron*, the device that forces charged particles to move in a circular path when subjected to a magnetic field. Electrons, protons, and ions are among the charged particles spun in cyclotrons. We will determine the fundamental frequency of the cyclotron by using some basic results from electromagnetism. A charged

Why?

How?

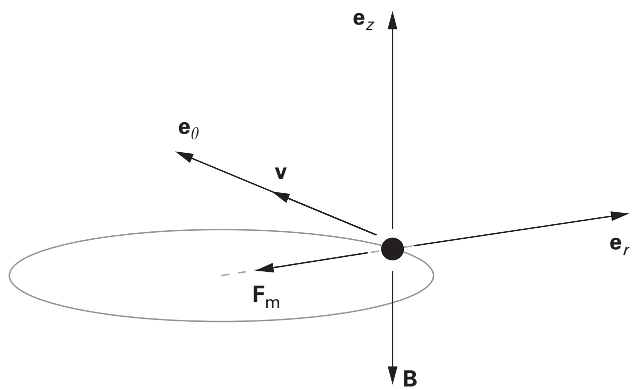


Figure 8.6 The cylindrical coordinate system and the basic vector structure needed to portray the elementary *cyclotron*. The coordinate system has the radial, tangential, and vertical unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$, respectively. The particle location is given by $\mathbf{r} = |\mathbf{r}|\mathbf{e}_r$. The magnetic field is directed in the $-z$ direction, that is, $\mathbf{B} = -|\mathbf{B}|\mathbf{e}_z$, and the magnetic force exerted on the charged particle is $\mathbf{F}_m$.

particle moving through a magnetic field is subjected to a magnetic force (vector), $\mathbf{F}_m$, given by:

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}, \tag{8.34}$$

where **B** is the *magnetic induction* (vector) due to currents other than that produced by the particle charge of magnitude q, **v** is the velocity (vector) of the moving charged particle, and the symbol $\times$ denotes the *vector* or *cross product* of the **v** and **B** vectors.

The geometry underlying our cyclotron model is shown in Figure 8.6. The particle motion is described in a cylindrical set of coordinates having radial, tangential, and vertical unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$, respectively. The location of the particle is given by $\mathbf{r} = |\mathbf{r}|\mathbf{e}_r$. The magnetic field is directed in the $-z$ direction, that is, $\mathbf{B} = -|\mathbf{B}|\mathbf{e}_z$. Thus, the vector equation (8.34) can be written as

$$\mathbf{F}_m = q\mathbf{v} \times (-|\mathbf{B}|\mathbf{e}_z), \tag{8.35}$$

where will soon identify the angle between the **v** and **B** vectors as θ.

Equations (8.34) and (8.35) indicate that the magnetic force, $\mathbf{F}_m$, is perpendicular to the particle velocity, **v**. Thus, the magnetic field, **B**, imparts no power to the particle. Further, since both the force, $\mathbf{F}_m$, and its consequent particle acceleration are perpendicular to the velocity, **v**, the particle must be traveling in a circle of radius, $|\mathbf{r}|$, at a (radian) frequency, ω_0, that also corresponds to simple harmonic motion. Further, that circular harmonic motion also means that the velocity vector is simply $\mathbf{v} = |\mathbf{v}|\mathbf{e}_\theta = |\mathbf{r}|\omega_0\mathbf{e}_\theta$ (see Problems 8.17–8.19). It then follows that eq. (8.35) becomes:

$$\mathbf{F}_\mathrm{m} = q(|\mathbf{v}|\mathbf{e}_\theta) \times (-|\mathbf{B}|\mathbf{e}_z) = q(|\mathbf{r}|\omega_0\mathbf{e}_\theta) \times (-|\mathbf{B}|\mathbf{e}_z) = -q|\mathbf{r}|\omega_0|\mathbf{B}|\mathbf{e}_r. \tag{8.36}$$

Equation (8.36) shows that the force, $\mathbf{F}_\mathrm{m}$, is directed radially inward, so that the acceleration is centripetal and also directed radially inward. Thus, just as with the centripetal acceleration of the pendulum (see eqs. (7.7a) and (7.8a)), the centripetal acceleration of the cyclotron particle is $-|\mathbf{r}|\omega_0^2\mathbf{e}_r$. Then, with the net force being $\mathbf{F}_\mathrm{m}$ of eq. (8.36), Newton's second law in the radial direction appears as

$$\mathbf{F}_m = -q|\mathbf{r}|\omega_0|\mathbf{B}|\mathbf{e}_r = -m|\mathbf{r}|\omega_0^2\mathbf{e}_r,$$

which finally yields the *cyclotron frequency*,

$$\omega_0 = \frac{q}{m}|\mathbf{B}|. \tag{8.37}$$

Equation (8.37) shows that the frequency depends only on the strength of the magnetic field, $\mathbf{B}$, and the charge-to-mass ratio, q/m, of the particle. It is independent of the radius of the circle and, therefore, the tangential velocity. Again, eq. (8.37) is the fundamental relationship behind cyclotron design.

Problem 8.14. If the fundamental dimension of charge is Q, determine the dimensions of the magnetic field or *magnetic flux density* $\mathbf{B}$ that ensure that eq. (8.34) is dimensionally correct.

Problem 8.15. The magnetic field $\mathbf{B}$ has units of *webers per square meter* (Wb/m^2) in SI units. Using eq. (8.34), express these units in terms of units of charge (the *coulomb*, C) and other fundamental dimensions in SI units.

Problem 8.16. Verify that the cyclotron frequency as given in eq. (8.37) is dimensionally correct.

Problem 8.17. Calculate the velocity components of a point located in a plane by the relation $\mathbf{r} = |\mathbf{r}|\mathbf{e}^{j\omega_0 t} = x(t)\mathbf{i}_x + y(t)\mathbf{i}_y$. Express that velocity in terms of (a) the time derivatives of $x(t)$ and $y(t)$ and then (b) in terms of $|\mathbf{r}|e^{j\omega_0 t}$.

Problem 8.18. Why do the results of Problem 8.17 express simple harmonic motion?

Problem 8.19. Calculate the velocity components of a point located in a plane by the relation $\mathbf{r} = |\mathbf{r}|\mathbf{e}_r$ and express the results in plane polar coordinates. How does this result compare with that found in Problem 8.17?

8.4 The Fundamental Frequency of an Acoustic Resonator

What is an acoustic resonator? We have all blown air across the top of a bottle and heard a deep, foghorn-like response. In fact, the frequency (or *pitch*) that we hear is very much a function of the size of the air cavity in the bottle (and *not* a function of the kind of liquid in the bottle!). An *acoustic resonator* is a flask or bottle with an air cavity that is used to produce sound. Such resonators are also called *Helmholtz resonators* after the German physicist who investigated it, Herman von Helmholtz (1821–1894). By what mechanism do acoustic resonators work?

Why?

How? We will answer that equation by modeling the flask shown in Figure 8.7 and, in so doing, we will account for the mechanics and thermodynamics of the changes in pressure and volume of a gas as it transmits a sound signal. The flask has an "interior" cavity of volume V_0, that contains a gas of density ρ_0, at ambient pressure, p_0. The neck of the flask is of length L and has a cross-sectional area A. We will see that the gas in the flask neck moves like a mass and that the cavity exerts a spring-like response to that movement, so that our resonator model will be a mass-spring system.

Predict?

Assume? We take the mass of gas in the neck as our mass, $m = \rho_0 AL$, to develop this model (or *this* analogy). The stiffness in the system comes from the gas in the cavity that resists being compressed as the neck mass moves toward it. That resistance is transmitted at the interface between neck and cavity by the pressure, p_0. The pressure, p_0, and the cavity volume of gas, V_0, that contains it are assumed to obey the *adiabatic gas law*:

$$pV^{\gamma} = \text{constant}, \tag{8.38}$$

where in this instance, γ is the ratio of heat capacities ($\gamma = 1.4$ for air, for example), and p and V are, respectively, pressure and volume. When the mass of gas in the neck, m, moves a distance, x, to the right, the cavity volume must be reduced by

$$\delta V = -Ax. \tag{8.39}$$

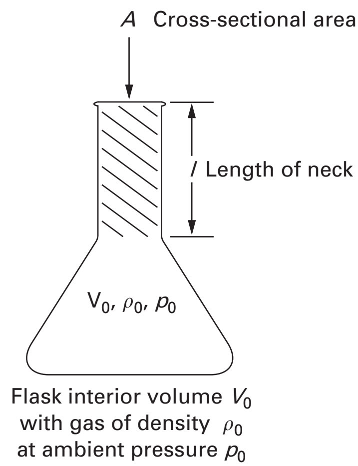


Figure 8.7 The flask used to model the acoustic or *Hemlholtz resonator*. The flask has a neck of length L, with area A, that is connected to an acoustic cavity of volume, V_0. The cavity contains a gas of density ρ_0, at pressure, p_0. When the mass in the neck moves, the cavity responds like a spring.

A small change of volume, δV, is related to a small change of pressure, δp, by the gas law (eq. (8.38)),

$$\delta(pV^\gamma) = V^\gamma(\delta p) + p(\gamma V^{\gamma-1}\delta V) = 0,$$

which, after dividing through by pV^γ, becomes

$$\frac{\delta p}{p} + \gamma\frac{\delta V}{V} = 0. \tag{8.40}$$

We now let the pressure and volume take on their ambient values, so eq. (8.40) becomes

$$\delta p = -\gamma\frac{p_0}{V_0}\delta V = 0. \tag{8.41}$$

Finally, we substitute the volume change, δV, from eq. (8.39) to find that the pressure change, δp is related to the distance moved by the neck mass, x, according to:

$$\delta p = \gamma p_0\frac{Ax}{V_0}. \tag{8.42}$$

Note that the dependence of δp on x is strongly suggestive of spring-like action, but the dimensions certainly don't look like those of a spring. On the other hand, if we recognize that the cavity-produced restoring force, F_{cavity}, acting on the neck mass is the product of pressure times area, then we see that

$$F_{\text{cavity}} = \delta p A = \gamma \frac{p_0 A^2}{V_0} x. \tag{8.43}$$

Now the resemblance to the classic spring is more evident (see Problem 8.20).

Then, if we blow across the open end of the flask with a force, $F(t)$, the mass, m, is pushed down the neck a distance, x, toward the cavity, and the cavity pushes back with a spring stiffness, k_{cavity}, the acoustic resonator behaves as a spring-mass system:

$$\rho_0 AL \frac{d^2x(t)}{dt^2} + \gamma \frac{p_0 A^2}{V_0} x(t) = F(t). \tag{8.44}$$

We can rewrite eq. (8.44) in terms of a parameter that is often used in acoustics and vibration problems, the *speed of sound* of the gas in the flask, c_0. That speed is related to the specific heat capacity, ambient pressure, and density of the gas:

$$c_0^2 = \gamma \frac{p_0}{\rho_0}. \tag{8.45}$$

Then the oscillator equation for the Helmholtz resonator is

$$\rho_0 AL \frac{d^2x(t)}{dt^2} + \frac{\rho_0 c_0^2 A^2}{V_0} x(t) = F(t). \tag{8.46}$$

The natural frequency or fundamental period follows from the homogenous version of the equation of motion (8.46) for the acoustic resonator (see Problem 8.22):

$$\omega_0 = \frac{2\pi}{T_0} = c_0 \sqrt{\frac{A}{V_0 L}}. \tag{8.47}$$

Predict? **Use?** Equation (8.47) could be accepted as the final result. It has the correct dimensions and shows that the frequency increases with the neck area but decreases as the neck gets longer and the cavity volume gets larger, which effects are consistent with our intuition (see Problems 8.23–8.26). **Improve?** However, a bit of reflection suggests that eq. (8.47) can be further massaged by identifying the volume of the neck (which is also the volume of the moving mass m) as $V_n = AL$. Then the frequency and period become:

$$\omega_0 = \frac{2\pi}{T_0} = \frac{c_0}{L} \sqrt{\frac{V_n}{V_0}}. \tag{8.48}$$

This version of the natural frequency of the acoustic resonator is even more interesting (and satisfying) because it shows the dependencies in a more meaningful way. The frequency goes down if we elongate the neck because it takes the mass longer to move down the neck, as we see from the ratio c_0/L. Further, the effect of increasing flask volume to get deeper (lower) frequencies will not be seen unless that volume reduction is done with respect to the neck volume. **Verified?**

Finally, the inhomogeneous version of the resonator model, eq. (8.46), begins to set the stage for the rest of this chapter. What does happen when there is a forcing function $F(t)$? What does $F(t)$ look like? It is easy enough to imagine that the wind blowing across the top is an acoustic signal that is, like most sounds, composed of many frequencies. Since eq. (8.46) is linear, we could obtain a complete solution by solving it for each frequency represented in $F(t)$ and then superposing or adding all of those solutions. This suggests that we seek a generic solution to **Why?** **How?**

$$\frac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = \frac{F_0}{\rho_0 V_{\text{neck}}} \cos \omega t. \tag{8.49}$$

The radian frequency, ω, in eq. (8.49) is arbitrary and can assume any value, so the forcing function represents any oscillatory signal or input. As we will see in Sections 8.6 and 8.7, there are some very interesting effects that occur. But, first, we want to explore another way in which forcing functions occur in models of vibration (Section 8.5), and then we will talk about the mathematics (Section 8.6) and the physics (Section 8.7) that occur in governing equations like eq. (8.49).

Problem 8.20. Show that the dimensions of $\gamma(p_0 A^2/V_0)$ are such that eq. (8.43) identifies the stiffness of the flask cavity, k_{cavity}.

Problem 8.21. How does the stiffness of a cavity change if the gas is assumed to be governed by the *ideal gas law*, $pV = nRT$?

Problem 8.22. Show that the homogeneous solution of eq. (8.46) requires that the resonator's natural frequency must be given by eq. (8.47). (*Hint*: Recall Section 7.2.2.)

Problem 8.23. Estimate the natural frequency of the cavity of a standard (750 ml) wine bottle. How does that frequency compare with the note middle C, for which $f = 262$ Hz?

Problem 8.24. How long would the wine bottle flask have to be to get its cavity frequency *below* the *lowest* note produced by a piano ($\sim$ 55 Hz)?

Problem 8.25. How long would the wine bottle flask have to be to get its cavity frequency *above* the *highest* note produced by a piano ($\sim$ 8360 Hz)?

Problem 8.26. Assume that a set of acoustic resonators is built like wine bottles, each with neck radius r_n, neck length L_n, cavity radius r_0, and cavity length L_0. How would the ratio L_n/L_0 vary with the radii if every bottle were to have the same natural period?

8.5 Forcing Vibration: Modeling an Automobile Suspension

Why? We finished our discussion of the acoustic resonator by noting how it could be forced to vibrate or respond, in that case with an excitation that was external and obvious. However, excitation can show up in models in **How?** other ways, as we now illustrate. Consider the damped oscillator shown in Figure 8.8(a) that is no longer connected to a fixed point or wall; rather, its free end travels over a specified contour, $y(z)$. It is a schematic for the suspension systems we are accustomed to seeing in cars, for example, and nowadays on high-end bikes. For the auto, the mass is that of the body, the power train, and the passengers and cargo. The spring is typically a coil spring that is wrapped around the shock absorber or damper. There was a time when auto springs were leaf springs, but their suspension systems would have been modeled the same way. The important feature is that both leaf and coil springs share common connection points with the shock absorber on the auto frame at one end and on the wheel at the other. Thus, spring and damper are in parallel with the auto's mass.

One way to set an auto suspension system in motion is to push rhythmically on its fender, a fairly common qualitative test of whether the shock absorbers retain much damping. This might be modeled in the same way we proposed modeling blowing over an acoustic cavity by including a forcing function, $F(t)$. In addition, however, the suspension system is excited or driven by the end connected to the wheel as it follows the road, $y(z)$. The model for the auto following the road contour is shown in Figure 8.8(b), where $a(t)$ is the amount that the wheel-end of the suspension moves with respect to a fixed wall. This means that the net extension of the spring is

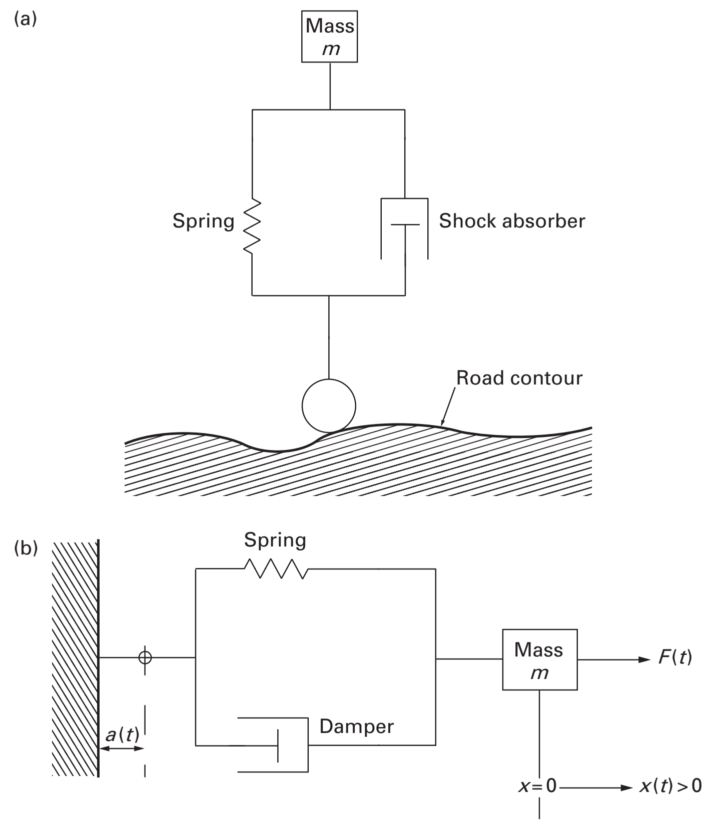


Figure 8.8 The spring-mass-damper system used to model the behavior of a *vehicle suspension system*: (a) the system's three elements (m, k, c) act in parallel and share the single coordinate, $x(t)$, while the other ends of the spring and damper share the wheel as a common connection point that follows the road contour, $y(z)$; and (b) the revision of the model to show the road-following wheel motion as a support that moves a distance, $a(t)$, with respect to the "traditional" spring-mass-damper.

$x(t) - a(t)$, and that the relative speed to which the damper responds is $d[x(t)-a(t)]/dt$. The spring force is then $k[x(t)-a(t)]$, and the damping force is $cd[x(t) - a(t)]/dt$, so that Newton's second law for this model is:

$$m\frac{d^2x(t)}{dt^2} = F(t) - k[x(t) - a(t)] - c\frac{d[x(t) - a(t)]}{dt},$$

or

$$m\ddot{x} + c\dot{x}(t) + kx(t) = F(t) + c\dot{a}(t) + ka(t). \tag{8.50}$$

Equation (8.50) shows that the terms due to the wheel motion, $a(t)$, remain on the right-hand side, because they are a known input. Thus, eq. (8.50) represents an instance of *forced* vibration even absent an explicit forcing function, that is, even when $F(t) = 0$.

Given? Consider the case of an auto without an explicit forcing function (i.e., with $F(t) = 0$) traveling in the z direction along a road whose contour $y(z)$ is given as:

$$y(z) = a_0 \sin \alpha z, \tag{8.51}$$

where α is a parameter with dimensions of $(\text{length})^{-1}$. If the auto moves down the road at constant speed, v, it follows that $z = vt$, so that the wheel motion is

$$a(t) = y(z = vt) = a_0 \sin \alpha vt. \tag{8.52}$$

Then the governing equation for the traveling suspension system is found when eq. (8.52) is substituted into eq. (8.50):

$$m\frac{d^2x(t)}{dt^2} + c\dot{x}(t) + kx(t) = a_0(k \sin \alpha vt + c\alpha v \cos \alpha vt). \tag{8.53}$$

Thus, for this model, we once again have a non-zero right-hand side or forcing function made up of trigonometric terms. And, again, this resulted not from an explicit external forcing function, but from the fact that the system's spring and damper were not attached to an immovable point.

Problem 8.27. What are the physical dimensions of the term αv in eq. (8.53)? Explain whether or not those dimensions are correct.

Problem 8.28. Determine the values of C_1 and ϕ that allow the right-hand side of eq. (8.53) to be written in the form $a_0 C_1(\cos(\alpha vt - \phi))$.

Problem 8.29. Determine the values of C_2 and ϕ that allow the right-hand side of eq. (8.53) to be written in the form $a_0 C_2(\sin(\alpha vt + \phi))$.

8.6 The Differential Equation $md^2x/dt^2 + kx = F(t)$

How? How do we determine the solution to the inhomogeneous differential equation that describes the dynamic response of an ideal, undamped oscillator that is driven by a harmonic forcing function (see Problems 8.28 and 8.29 above):

$$m\frac{d^2x(t)}{dt^2} + kx(t) = F_0 \cos(\omega t - \phi). \tag{8.54}$$

The solutions to inhomogeneous differential equations have two parts that are superposed. The first part is the transient solution to the homogeneous equation that we had already found as eq. (7.48) or (7.49) in Section 7.2.2. The second part is the *particular* or *steady-state solution* that is crafted to solve only the differential equation without regard to the system's initial conditions.

As a trial particular solution let us assume that

$$x(t) = X_0 \cos(\omega t - \phi), \tag{8.55}$$

where X_0 is a constant yet to be determined. By direct substitution of eq. (8.55) into eq. (8.54), we get:

$$(k - m\omega^2)X_0 \cos(\omega t - \phi) = F_0 \cos(\omega t - \phi),$$

which means that

$$X_0 = \frac{F_0}{k - m\omega^2} = \frac{F_0/k}{1 - (\omega/\omega_0)^2}, \tag{8.56}$$

where once again ω_0 is the natural frequency of the ideal oscillator defined in eq. (8.7). The final form of the steady-state solution is, then,

$$x(t) = \frac{F_0/k}{1 - (\omega/\omega_0)^2} \cos(\omega t - \phi). \tag{8.57}$$

This all seems perfectly straightforward but for one detail: If the frequency of the driving force, ω, happens to equal the natural resonance of the system, ω_0, the solution (8.57) "blows up" or becomes infinite. Now in the real world that may not literally happen because of damping, but even with the ameliorating effect of damping there is a problem when $\omega = \omega_0$. In the next section we will identify that as *resonance*, but here we want to stay focused on the formal mathematics. To complete that we note simply that a formal solution to eq. (8.54) does exist for the case $\omega = \omega_0$, and that solution can be shown to be (see Problem 8.31):

$$x(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t - \phi). \tag{8.58}$$

Note that $x(t)$ depends linearly on t in eq. (8.58), a result that clearly confirms the singular behavior of the ideal spring-mass system when it is excited or driven at its natural frequency. In the real world, again, damping comes very much into play, and avoiding such resonant behavior (even with damping) is a major priority in the design of vibrating systems. We will have more to say about that in Section 8.7.

Problem 8.30. Determine the value of X_0 in eq. (8.55) by substituting eq. (8.55) into eq. (8.54) and ensuring that the equation of motion is indeed satisfied.

Problem 8.31. Confirm that the solution (8.58) does satisfy eq. (8.54) for the special case of resonance, that is, when $\omega = \omega_0$.

Problem 8.32. Determine and explain the dimensions of the coefficients $(F_0/m\omega_0)$ in eq. (8.58).

8.7 Resonance and Impedance in Forced Vibration

We now turn to the meaning and physical implications of the mathematics of simple forced oscillators. So, we again start with the equation of motion of an ideal spring-mass system that is driven by a harmonic excitation:

$$m\frac{d^2x(t)}{dt^2} + kx(t) = F_0 \cos(\omega t - \phi). \tag{8.59}$$

The complete solution to eq. (8.59) is the sum of the homogeneous or transient solution (7.48) and the particular or steady-state solution (8.57):

$$x(t) = B_1 \cos \omega_0 t + B_2 \sin \omega_0 t + \frac{F_0/k}{1 - (\omega/\omega_0)^2} \cos(\omega t - \phi). \tag{8.60}$$

where B_1 and B_2 are arbitrary constants that will be determined by the initial conditions set for the system. Having written the complete solution, it must be said that our primary interest lies in the steady-state solution because it predicts the behavior of the spring-mass system for as long as we drive it with the harmonically varying force in eq. (8.59). Further, it is independent of the initial conditions, which, as we noted in Section 7.22, affect only the transient behavior. (It should be noted that the notion of a transient solution that, implicitly, does not affect the steady state, does assume that there is at least a little bit of damping, so that solutions initiated only by the initial conditions will die out. The steady-state solution persists even in the face of damping because the excitation persists.)

Since we can always incorporate the effects of the initial conditions by suitably adjusting the two arbitrary constants in the complete solution (8.60), we take eq. (8.60) in the following form as the solution of interest:

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t - \phi). \tag{8.61}$$

We note that $x(t)$ has the same temporal behavior as the forcing function, that is, its behavior in time is the same. Thus, we say that the motion of the mass is *in phase* with the action of the driver. On the other hand, the speed of the response is given by

$$\frac{dx(t)}{dt} = -\frac{\omega F_0}{m(\omega_0^2 - \omega^2)} \sin(\omega t - \phi), \tag{8.62}$$

which shows that the speed is out of phase with the driver by 90°, that is, the speed of the mass lags behind the force by a time equal to $t = \pi/2\omega$. Now eq. (8.62) can also be written as (see Problem 8.35):

$$\frac{dx(t)}{dt} = \frac{F_0}{m\omega_0[(\omega/\omega_0) - (\omega_0/\omega)]} \sin(\omega t - \phi), \tag{8.63}$$

As we just saw in Section 8.6, the displacement and the speed become infinitely large as the forcing frequency, ω, approaches the natural frequency, ω_0. Thus, when the driving frequency equals the natural frequency, we have the condition of *resonance*. The oscillatory forcing function produces an infinite response. In fact, resonance is what we are trying to achieve when we time the pushes given to someone seated on a playground swing! In Figure 8.9 we have sketched the shape of the ideal response curve of eqs. (8.61) or (8.57) on a set of axes rendered dimensionless: $kx(t)/F_0$ against ω/ω_0. The infinite peak for the ideal case is quite obvious. We have also shown there a sketch of the damped response, which we will discuss shortly, but note that it is bounded and finite.

In acoustics and vibration research and practice, resonance and other vibratory phenomena are exhibited and measured in terms of a system's impedance, which for the system modeled here is:

$$Z(\omega) \equiv |F(t)| \Big/ \left|\frac{dx(t)}{dt}\right|,$$

which means that the *impedance for an ideal spring-mass system* is

$$Z(\omega) = m\omega_0 \left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right). \tag{8.64}$$

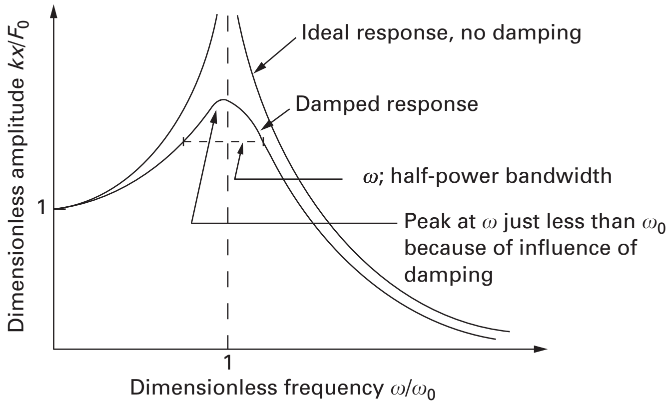


Figure 8.9 A sketch of the shape of the ideal response curve of a spring-mass system driven by a harmonic excitation. The axes are dimensionless: $kx(t)/F_0$ against ω/ω_0. The infinite peak for the ideal case is quite obvious. The damped response is bounded and finite.

We see in eq. (8.64) that the impedance vanishes at resonance, that is, $Z(\omega_0) = 0$. Thus, when the speed of the mass becomes infinite, nothing impedes its motion—even if the magnitude of the force is very small. Thus, an alternate statement of the condition of resonance is that it occurs at the frequency for which the impedance $Z(\omega) = 0$ vanishes.

The form of eq. (8.65) also suggests that the behavior of $Z(\omega)$ might be substantially different for $\omega < \omega_0$ than it would be for $\omega > \omega_0$. In fact, for frequencies below the natural frequency (i.e., for $\omega \ll \omega_0$), eq. (8.64) can be approximated as

$$Z_{\mathrm{k}}(\omega) \cong \frac{m\omega_0^2}{\omega} = \frac{k}{\omega}. \tag{8.65}$$

Thus, for low frequencies, where the excitation is applied slowly, the oscillator responds as a spring: The impedance decreases as the frequency increases toward the natural frequency. For low frequencies, of course, we are closer to the static limit of $\omega = 0$, so it should not be a surprise that stiffness dominates the response.

On the other hand, for frequencies above the natural frequency (i.e., for $\omega \gg \omega_0$), eq. (8.64) can be approximated as

$$Z_{\mathrm{m}}(\omega) \cong -m\omega. \tag{8.66}$$

At high frequencies we expect the dynamics to be more important, and so it is not unexpected that the mass dominates. It is also not surprising that the impedance increases with frequency, meaning that it gets progressively harder to push around a mass at ever-higher frequencies.

So much for the ideal case. What happens in the "real world" where there is friction and damping and energy loss? The mathematics of modeling damped systems get more complex (see Problems 8.37 and 8.38), so we will present a few key results here. The governing equation for analyzing the dynamic response of a damped oscillator is:

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = F_0 e^{j\omega t}. \tag{8.67}$$

A damping element is included here, and we also have introduced complex arithmetic in the notation for the excitation: The forcing function is written in exponential form (see Sections 4.9 and 7.2.2) and, in order that eq. (8.67) remain real, the forcing amplitude must be a complex number. It can be shown that the square of the magnitude of the resulting motion of a spring-mass-damper is:

$$|x(t)|^2 = \frac{|F_0|^2}{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}, \tag{8.68}$$

while the magnitude of the impedance is:

$$|Z(\omega)|^2 = m^2\omega_0^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + c^2. \tag{8.69}$$

We note immediately that eqs. (8.68) and (8.69) reduce to their respective counterparts for the ideal model (eqs. (8.61) and (8.64)) when $c = 0$. Further, and still more important, note that the presence of damping eliminates both the singular response and the vanishing of the impedance at resonance. Thus, at resonance, when $\omega = \omega_0$,

$$|x(t)|^2_{\omega_0} = \frac{|F_0|^2}{c^2\omega_0^2}, \tag{8.70}$$

and

$$|Z(\omega_0)|^2 = c^2. \tag{8.71}$$

Equation (8.70) shows that the response is bounded and non-infinite as long as there is damping, and that it becomes infinite when $c = 0$. Equation (8.71) shows that the impedance vanishes altogether only if the damping vanishes altogether. In fact, eqs. (8.69) and (8.71) both also confirm our intuitive sense that damping impedes motion.

Problem 8.33. What are the fundamental physical dimensions of impedance for a mechanical oscillator?

Problem 8.34. Show that the mechanical impedance of an ideal spring-mass system can be written in the form

$$Z(\omega) = \frac{k}{\omega} - m\omega.$$

Explain why this form of impedance does not work as well as eq. (8.64) to discern the stiffness- and mass-controlled regions of response.

Problem 8.35. Write the governing equation for a parallel *LC* circuit subject to a harmonic current input $-(i_0/\omega)\cos(\omega t - \phi)$ and determine the resulting impedance.

Problem 8.36. What are the fundamental physical dimensions of impedance for an electrical oscillator? [*Hints*: Imagine eq. (8.71) and its predecessor with a resistor, *R*, in place of the damping coefficient, or solve Problem 8.35.]

Problem 8.37. Assume an exponential solution to the homogeneous counterpart of eq. (8.67) and determine the roots for which a solution exist.

Problem 8.38. Determine the particular solution to eq. (8.67) by assuming that $x(t) = B\exp(j\omega t)$, where B and ω may be complex.

Problem 8.39. Determine and explain the dimensions of the coefficients, $(F_0/m\omega_0)$, in eq. (8.58).

Problem 8.40. Sketch the impedance, $Z(\omega)$, of a spring-mass-damper against the dimensionless frequency and identify the regimes where stiffness, mass, or damping controls the response.

8.8 Summary

We have devoted this chapter to the simple harmonic oscillator, without and with damping, without and with a forcing function, and in several different guises. These applications have included the classical mechanical spring-mass system, inductor-capacitor oscillators, a parallel *RLC* circuit, the vibration of tall buildings, and oscillation in a cyclotron and of a vehicle suspension system. We also developed the electrical-mechanical analogy and pointed out its usefulness for thinking about the meaning of the different terms in the various oscillator models.

In addition, we solved the differential equation and described the solution for the forced harmonic vibration of an oscillator. In so doing, we were able to bring out the very important concepts of resonance and impedance. In discussing impedance, we showed how the various elements (spring, mass, and damper) provided different response regimes, that is, frequency regimes that are controlled, respectively, by stiffness, mass, and damping.

And, finally, we pointed out the commonality of both the mathematics and the physics of such system models. Thus, to develop oscillatory behavior, systems must have elements with stiffness that store potential energy (springs and capacitors) elements with mass that store kinetic energy (masses and inductors), and elements that dissipate energy (dashpots and resistors). Stiffness may take many forms, but there must always be an element that stores potential energy in order for there to be an exchange with an element that stores kinetic energy.

8.9 References

D. P. Billington, *The Tower and the Bridge: The New Art of Structural Engineering*, Basic Books, New York, 1983.

R. E. D. Bishop, *Vibration*, Cambridge University Press, Cambridge, UK, 1965.

R. E. D. Bishop and D. C. Johnson, *The Mechanics of Vibration*, Cambridge University Press, Cambridge, UK, 1960.

M. Braun, *Differential Equations and Their Applications: Shorter Version*, Springer-Verlag, New York, 1978.

R. H. Cannon, Jr., *Dynamics of Physical Systems*, McGraw-Hill, New York, 1967.

P. D. Cha, J. J. Rosenberg, and C. L. Dym, *Fundamentals of Modeling and Analyzing Engineering Systems*, Cambridge University Press, New York, 2000.

C. L. Dym and E. S. Ivey, *Principles of Mathematical Modeling*, 1st Edition, Academic Press, New York, 1980.

C. L. Dym and I. H. Shames, *Solid Mechanics: A Variational Approach*, McGraw-Hill, New York, 1973.

F. Fahy, *Sound and Structural Vibration*, Academic Press, London, 1985.

M. Farkas, *Dynamical Models in Biology*, Academic Press, San Diego, CA, 2001.

R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vols. I and II, Addison-Wesley, Reading, MA, 1963.

B. R. Gossick, *Hamilton's Principle and Physical Systems*, Academic Press, New York, 1967.

R. Haberman, *Mathematical Models*, Prentice-Hall, Englewood Cliffs, NJ, 1977.

D. Halliday and R. Resnick, *Fundamentals of Physics*, 2nd Edition, Revised Version, John Wiley & Sons, New York, 1986.

G. W. Housner and D. E. Hudson, *Applied Mechanics: Dynamics*, Von Nostrand-Reinhold, New York, 1959.

E. C. Pielou, *An Introduction to Mathematical Ecology*, Wiley Interscience, New York, 1969.

A. D. Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications*, McGraw-Hill, New York, 1981.

J. W. S. Rayleigh, *The Theory of Sound*, Vol. 1, 2nd Edition, Dover Publications, New York, 1945. (The 1st edition was published in 1877.)

E. Simiu and R. H. Scanlan, *Wind Effects on Structures*, John Wiley & Sons, New York, 1978.

J. M. Smith, *Mathematical Ideas in Biology*, Cambridge University Press, London and New York, 1968.

G. W. Swenson, *Principles of Modern Acoustics*, Boston Technical Publishers, Cambridge, MA, 1965.

B. S. Taranath, *Structural Analysis and Design of Tall Buildings*, McGraw-Hill, New York, 1988.

P. A. Tipler, *Physics*, Worth Publishers, New York, 1976.

J. W. Tongue, *Principles of Vibration*, 2nd Edition, Oxford University Press, New York, 2001.

M. R. Wehr and J. A. Richards, Jr., *Physics of the Atom*, Addison-Wesley, Reading, MA, 1960.

R. M. Whitmer, *Electromagnetics*, Prentice-Hall, Englewood Cliffs, NJ, 1962.

H. B. Woolf (Editor), *Webster's New Collegiate Dictionary*, G. & C. Merriam, Springfield, MA, 1977.

8.10 Problems

8.41. The height of a World Trade Center (WTC) tower was 1370 ft (110 stories) and its fundamental period was about 11 sec. The height of the Empire State Building is 1250 ft (102 stories) and its fundamental period is about 8 sec.

(a) How do their respective average specific weights compare?

(b) If the average specific weight for the WTC is as given in Section 8.2 for slender steel-framed towers, what would be the corresponding number for the Empire State Building?

8.42. The height of the Citicorp Building is 915 ft (59 stories) and its fundamental period is about 6.5 sec. Given the data in Problem 8.41 for a WTC tower, find:

(a) how the period varies with building height; and
(b) how the period varies with number of stories.

8.43. Obtain an expression [analogous to eq. (7.28)] for the total energy in a parallel *RLC* circuit and calculate its rate of change with respect to time [analogous to eq. (7.29)].

8.44. Obtain an approximate expression [analogous to eq. (7.30)] for the total energy in a parallel *RLC* circuit that can be used with the results of Problem 8.43 to obtain a differential equation [analogous to eq. (7.29)] for the circuit's energy.

8.45. Use the results of Problem 8.44 to determine how the energy of the parallel *RLC* circuit behaves over time? What is the relevant time constant, and how would you characterize that constant? (*Hint*: Reread Section 7.1.6.)

8.46. (a) Find the impedance of an acoustic resonator as a function of ρ_0, A, L, V_0 and ω; and
(b) What are the physical dimensions of the resonator impedance?

8.47. Charged particles are accelerated in a cyclotron travel in circles of radius r that depends on their speed, v, and magnetic flux density, B, according to:

$$r = \frac{mv}{qB},$$

where m and q are the particle's mass and charge, respectively. The speed and the energy are boosted every half-cycle, so that the particles execute forced harmonic motion in circles whose radii are increasing .

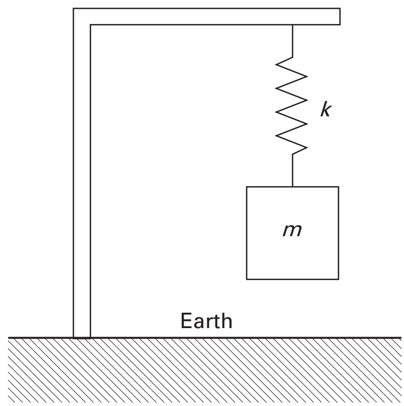
(a) At what resonant frequency ω_0 must the energy be supplied?
(b) What is the impedance of this system?
(c) Show that the rate of change of the energy in the system is of the form

$$\frac{dE(t)}{dt} = \frac{(qB)^3(r_2^2 - r_1^2)}{2\pi m^2} > 0.$$

8.48. A simple seismograph is shown in the accompanying figure. If y denotes the displacement of m relative to the earth, and η the displacement of the earth's surface relative to the fixed stars, the equation of motion of the mass is

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = -m\frac{d^2\eta(t)}{dt^2}.$$

(a) Determine the steady-state response if $\eta(t) = C\cos\omega t$.
(b) Sketch the amplitude of $y(t)$ as a function of ω.



8.49. What are the dimensions of the *damping quality factor*, $Q = \omega_0 m/c$?

8.50. A long-period seismometer has mass, $m = 0.01$ kg, period, $T_0 = 30$ sec, and damping quality factor, $Q = 3$. An earthquake triggers the earth's surface to respond with a oscillations with a period of 15 minutes and a maximum acceleration of 2×10^{-9} m/sec^2. What is the amplitude of the seismometer vibration?

8.51. Given that power equals force times velocity or speed, determine the *average power* needed to maintain the oscillations of a damped system driven by $F = F_0\cos\omega t$ and responding as $x(t) = X_0\cos(\omega t + \phi)$, where ϕ is the phase angle by which the response lags behind the force.

8.52. For the forced oscillator of Problem 8.51, let the phase angle $\phi = \pi/2$ rad, $\omega_0 = 500$ rad/sec, $Q = 4$ when

$$X_0 = \frac{F_0}{k}\frac{\omega_0/\omega}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \left(\frac{1}{Q}\right)^2}},$$

with $\sin\phi = Q(kX_0/F_0)(\omega/\omega_0)$.

(a) Plot the average power input found in Problem 8.51 against the frequency, ω, of the driving force.

(b) Find the width $\Delta\omega$ of the power curve of part (a) at one-half of the maximum power, centered around the resonance frequency. This range of frequencies, the *half-power band*, is that within which resonance effectively occurs.

8.53. (a) Repeat the calculations of Problem 8.52 with a damping quality factor $Q = 6$.
(b) What does a comparison of the two half-power bands for different values of Q reveal about the effect of damping on resonance?

8.54. List resonant systems that we see in nature, over as wide a range as possible.

8.55. A weight hanging on the end of a spring causes a *static deflection* $x_{st} = W/k$. If the static deflection is measured in inches, show that the resonant frequency in cycles per second is $f = 3.13/\sqrt{x_{st}}$ (Hz).

8.56. A bridge is 100 m long and supported by steel beams whose modulus of elasticity is $E = 2\,\text{N/m}^2$ and whose second moment $I = 0.002\,\text{m}^4$. Determine the bridge's natural frequency if its mass is 10^5 kg and a weight of 1.8×10^5 N causes it to deflect 0.01 m?

8.57. A group of 200 soldiers who collectively weigh 1.8×10^5 N marches in step across the bridge of Problem 8.56. Their right feet hit the bridge at regular intervals of 0.9 sec, forcing the bridge to vibrate. Would an observer see that vibration? Explain how you know that.