4
Approximating and Validating Models

We devote this last chapter on fundamentals to discussions of elementary mathematical approximation techniques and of model testing and validation. Approximations are used to simplify both models (as we will see in Chapter 7 where the nonlinear model of the pendulum is simplified to obtain a linear estimate of the pendulum's behavior) and the numerical calculations made with the models. Such approximations and their numerical implementations introduce *error*, but the magnitudes of these errors can be estimated and limited. We will also discuss means of model validation: checking dimensions and units, testing qualitative behavior and limits, and applying basic statistics.

4.1 Taylor's Formula

Engineering and scientific calculations abound with mathematical approximations, in some measure because linear problems are easier to solve, but in larger measure because many of our linear models are validated and justified by experiment and by experience. Distinctions such as those between a linearized model and its full nonlinear counterpart also involve mathematical approximations such as those described in this section. How do we approximate a function to properly estimate the behavior it describes?

Many analytical approximations are derived from Taylor's formulas. Advanced numerical techniques such as the finite element method also use Taylor's formulas to approximate functions as polynomials with unknown coefficients that are determined numerically. Thus, we now review some basic results about Taylor's formula and series, including Taylor formulas of trigonometric functions and binomial expansions.

4.1.1 Taylor's Formula and Series

Any function that is continuous and has derivatives can, in general, be expanded into and approximated by a Taylor's formula. For values of the independent variable, x, in a region near $x = a$, a function $f(x)$ can be approximated by the polynomial

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{4.1}$$

where $f'(a)$ represents the first derivative of $f(x)$, $f''(a)$ the second derivative, and $f^{(n)}(a)$ the nth derivative of $f(x)$ evaluated at the point $x = a$. The series given in eq. (4.1) is called the *Taylor formula of f(x) in the neighborhood of the point* $x = a$. The point $x = a$ must be such that all derivatives of $f(x)$ exist there and are finite. In addition, and most important for this discussion, if the difference $(x - a)$ is very small, then we need only a few terms of the series (4.1) to render a good approximation of $f(x)$ in the neighborhood of $x = a$. The corresponding *Taylor's series* that renders the approximate equality in eq. (4.1) an exact equality is the limit of eq. (4.1) as $n \to \infty$:

$$f(x) = \lim_{n\to\infty} \left[f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right]. \tag{4.2}$$

If we want to approximate the function $f(x)$ at another point, say $x = b$, we evaluate eq. (4.1) at that point to find Taylor's formula for $f(b)$:

$$f(b) \cong f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n. \tag{4.3}$$

If we use only the first term of eq. (4.3), we are approximating $f(b)$ as being equal to $f(a)$, as shown in Figure 4.1(a). If we use the first two terms of eq. (4.3), our approximation is improved by incorporating the effect of the slope change $f'(a)$, as shown in Figure 4.1(b). This value is closer to the true value than our simple one-term approximation. Our approximation is still further improved when three terms of the expansion (4.3) are used to approximate $f(b)$, as shown in Figure 4.1(c).

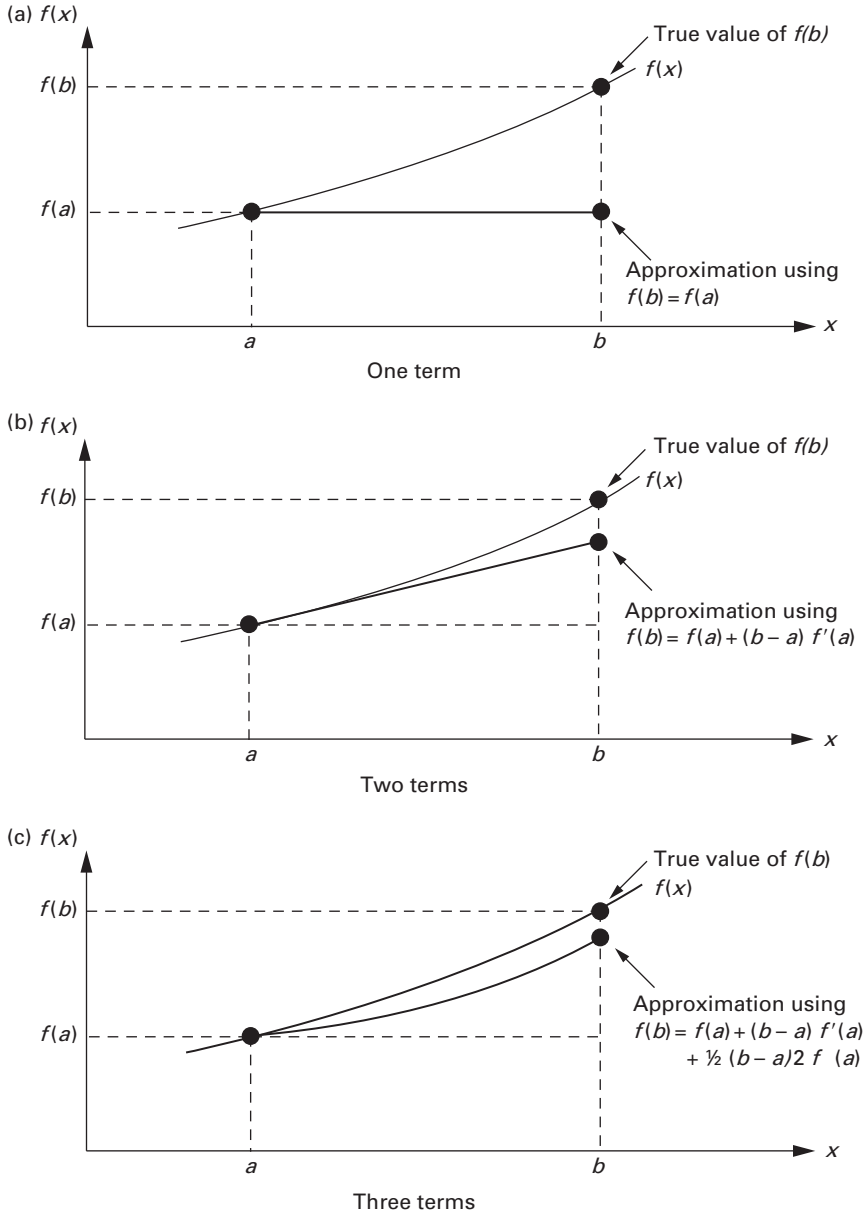

Figure 4.1 Improving the approximations obtained with a Taylor expansion by retaining more terms: (a) a one-term series approximation; (b) a two-term estimate; and (c) a three-term approximation. Note that the higher-order approximations depend on derivatives of $f(x)$ at the reference point of the Taylor series, $x = a$.

The accuracy of an approximation for any function $f(x)$ improves with the number of terms used in the expansion. Similarly, the approximation in eq. (4.1) can be turned into an exact formula like eq. (4.2) by adding a *remainder term* R_{n+1} to eq. (4.1):

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_{n+1}, \tag{4.4}$$

where the remainder term (which can be cast in several forms) is here shown as:

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-a)^{n+1}\,. \tag{4.5}$$

The derivative in eq. (4.5) is calculated at a "suitably chosen" point ξ somewhere in the interval between a and x. Even though the precise location of ξ is not known, the remainder formula can be used to estimate the error made if a Taylor formula to order n is applied (see Problem 4.31). How many terms do we have to keep in a Taylor formula to ensure that the error is negligible, or at least acceptable? As we will see below, it depends on what we're trying to do, on the specifics of the model we're trying to build.

4.1.2 Taylor Series of Trigonometric and Hyperbolic Functions

The Taylor series expansions of the trigonometric functions for $a = 0$ are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\,, \tag{4.6a}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\,. \tag{4.6b}$$

where x is expressed in (dimensionless) radians to ensure dimensional homogeneity. The corresponding Taylor expansions for the hyperbolic functions are:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\,, \tag{4.7a}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\,. \tag{4.7b}$$

We will now use a Taylor formula for the hyperbolic cosine (eq. (4.7b)) to estimate the sag of a tightly stretched string or cable that is weighted down only by its own weight. Such a cable is called a *catenary* after the

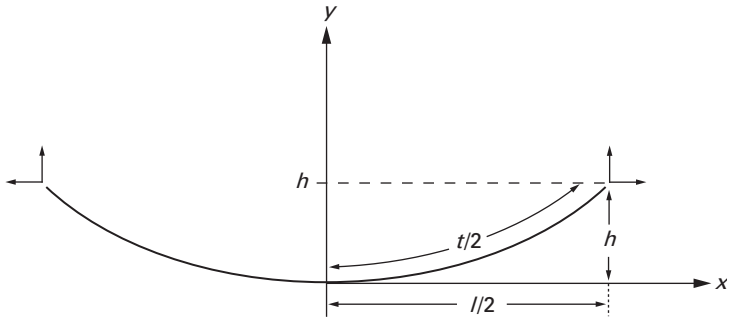

Figure 4.2 A long measurement tape stretched between two fixed points, A and B, for which the sag, h, is exaggerated. The mathematical model of the stretched tape is a hyperbolic cosine that can be approximated to varying degrees, depending on the relative magnitude of the ratio, h/l. This dependence signifies the fact that actual tape readings, t, must be corrected to properly measure the distance, l, on the ground.

Latin word for chain. Estimating the sag of a catenary may not sound all that interesting, but it does have a practical side that had been, until recently, a real engineering application. Until theodolites were introduced to measure large distances in construction projects, surveyors and engineers relied on tape measures. A surveyor's tape acts as a catenary because its only vertical load while measuring is its self-weight. We show such a tape in Figure 4.2, stretched between two supports at the same elevation that are separated by the length, l, with the cable's sag, h, exaggerated. Since $\cosh(0) = 1$, the equation of the catenary is

$$y(x) = c\left(\cosh\frac{x}{c} - 1\right), \tag{4.8}$$

where c is the *catenary parameter* and the coordinates of the *vertex* or low point of the cable are $(x = 0, y = c)$. The catenary parameter is a function of T_0, the (constant) horizontal component of the tension in the stretched cable, and of γ, the string's weight per unit length (see Problem 4.4). We see from Figure 4.2 that the sag is given by

$$h = y(l/2) = c\left(\cosh\frac{l}{2c} - 1\right). \tag{4.9}$$

Now we substitute the Taylor series (4.7b) of the hyperbolic cosine to find the sag:

$$h = c\cosh\frac{l}{2c} - c = c\left(1 + \frac{1}{2!}\frac{l^2}{4c^2} + \frac{1}{4!}\frac{l^4}{16c^4} + \frac{1}{6!}\frac{l^6}{64c^6} + \cdots - 1\right)$$
$$= c\left(\frac{1}{2!}\frac{l^2}{4c^2} + \frac{1}{4!}\frac{l^4}{16c^4} + \frac{1}{6!}\frac{l^6}{64c^6} + \cdots\right). \tag{4.10}$$

Note that this Taylor series for the sag has the correct physical dimensions since both c and h are measures of length and the ratio l/c is dimensionless, as it should be as the argument of the hyperbolic function. Further, for a tightly stretched string, the sag, h, is very small compared to the length, l, that is, $h/l \ll 1$. This suggests that the ratio $l/2c$ is also quite small compared to 1 because a one-term approximation of eq. (4.9) is found by retaining only the first term in the last of eq. (4.10):

$$h \cong c\left(\frac{1}{2!}\frac{l^2}{4c^2}\right) = \frac{l^2}{8c}. \tag{4.11}$$

Equation (4.11) confirms the suggestion that large values of the dimensionless catenary parameter, $2c/l$, correspond to small values of the dimensionless sag, h/l, because this result can be arranged as:

$$\frac{2c}{l} = \frac{l}{4h} \gg 1. \tag{4.12}$$

Further, had we approximated the hyperbolic cosine for small values of $l/2c$ independently of eqs. (4.9) and (4.10), we would have calculated that

$$c\cosh\frac{l}{2c} \cong c\left(1 + \frac{1}{2!}\frac{l^2}{4c^2} + \frac{1}{4!}\frac{l^4}{16c^4} + \frac{1}{6!}\frac{l^6}{64c^6}\right) \cong c, \tag{4.13}$$

and we would then have found, quite mistakenly, that the sag was identically zero because we had used an inadequate approximation!

How do these results affect the measurements of long distances with a tape? The answer is found by calculating the length of tape, t, needed to measure the horizontal distance, l, as shown in Figure 4.2. An element of arc length along the tape, ds, is given by

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + (y'(x))^2}. \tag{4.14}$$

If we substitute the catenary shape (4.8) into eq. (4.14) and apply a standard identity, we find that

$$ds = \cosh\frac{x}{c}\,dx. \tag{4.15}$$

Equation (4.15) can be straightforwardly integrated, that is,

$$\int_0^{t/2} ds = \int_0^{l/2} \cosh \frac{x}{c}\, dx,$$

to yield (see Section 4.8):

$$t = 2c \sinh \frac{l}{2c}. \tag{4.16}$$

We can expand eq. (4.16) in a Taylor formula, again based on the assumption that $l/2c$ is quite small, but for reasons that will soon become evident, we will retain the first two terms in the series, that is:

$$t \cong 2c \left(\frac{l}{2c} + \frac{1}{3!} \frac{l^3}{8c^3} \right). \tag{4.17}$$

With the aid of either eq. (4.11) or eq. (4.12), eq. (4.17) can be written as a quadratic equation in the distance l:

$$l^2 - lt + \frac{8}{3} h^2 = 0. \tag{4.18}$$

The quadratic equation (4.18) can be solved for its roots:

$$2l = t \left(1 \pm \sqrt{1 - \frac{32}{3} \left(\frac{h}{t} \right)^2} \right). \tag{4.19}$$

Only the positive root is physically viable here. In the next section, we will see that the radicand in eq. (4.19) is an ideal candidate to be written as a *binomial expansion*, which is a special form of Taylor's formula. For small values of h/l and to two term accuracy,

$$2l \cong t \left(1 + \left(1 - \frac{32}{6} \left(\frac{h}{t} \right)^2 \right) \right) = t \left(2 - \frac{32}{6} \left(\frac{h}{t} \right)^2 \right). \tag{4.20}$$

Thus, the actual length, l, that is measured by a tape reading of t is given by

$$l \cong t \left(1 - \frac{8}{3} \left(\frac{h}{t} \right)^2 \right). \tag{4.21}$$

Obviously, the larger the sag, h, the larger the correction that must be applied to the tape reading, t, to ensure an accurate measurement of the distance, l.

Lastly on the expansion (4.10), we point out that it is an approximation in the spirit of the *small angle* approximation that appears frequently in

engineering and scientific models. For example, from eq. (4.6b) we know that the second-order Taylor formula for the elementary cosine can be written as

$$\cos x \cong 1 - \frac{x^2}{2!}, \tag{4.22}$$

where x is measured in radians. To approximate the cosine function for very small angles in the neighborhood of $x = 0$, we can safely ignore the second-order term in eq. (4.22) and take $\cos x \cong 1$. However, as we will see in the formal development of the pendulum model in Chapter 7, we often have reason to approximate a slightly different function, $(1 - \cos x)$. If we neglected or ignored the second-order term here, the resulting approximation would be $(1 - \cos x) \cong 0$, which is a bad approximation that results from throwing out the dependence on x. Thus, as in so many other aspects of modeling, it is important to know where we're going when truncating Taylor formulas or series.

There is another approach to approximating trigonometric functions that is worth mentioning. Suppose we wanted to replace $\sin x$ by x in a model or a calculation. We could look at the numerical values of both functions to see where the substitution would be acceptable. For example, if we are willing to accept an error of 5%, we could replace $\sin x$ by x for $x \leq \pi/6$. For an error of only 2%, the substitution would be acceptable for $x \leq \pi/12$. (And while it is important that all angles in these arguments be either rendered as dimensionless ratios of variables or expressed as angles measured in radians, it is worth noting that the two examples just given correspond to small angles of, respectively, 30° and 15°.) Thus, by exploring the numerical ranges of interest and the associated errors, we can often justify replacing a trigonometric function by an algebraic approximation.

4.1.3 Binomial Expansions

Another Taylor series that is used often in engineering and science is the *binomial expansion*:

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots . \tag{4.23}$$

Equation (4.23) is valid for all values of n, and it converges for $x^2 < a^2$. Further, when n is a positive integer, the series (4.23) has only a finite number of terms.

Equation (4.23) is very useful in applications when x is rendered dimensionless with respect to a. (Recall that the principle of dimensional

homogeneity requires that x and a have the same physical dimensions.) If we divide eq. (4.23) by a^n, we find that

$$\left(1+\frac{x}{a}\right)^n = 1+n\left(\frac{x}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3+\cdots . \tag{4.24}$$

This is an ideal form for extracting expansions valid for values of $(x/a) \ll 1$.

We will illustrate the use of binomial expansions by looking at a familiar mechanics problem, the estimation of the weight of a mass, m, that is held at some height, h, above the surface of the earth. The weight, W, is the gravitational force, F_g, as expressed by *Newton's law of gravitational attraction*, which can be expressed in scalar form as:

$$F_g = -\frac{Gm_e m}{R^2} = -W, \tag{4.25}$$

where G is the universal gravitational constant, m_e the mass of the earth, and R is the distance between the centers of m and m_e. The minus sign in front of W follows because of the sign convention implied in eq. (4.25) wherein the gravitational force, F_g, would be positive directed away from the earth, while we would customarily draw W as a positive quantity (an arrow) directed toward the earth. Now, if we measure the distance to the mass, m, from the earth's surface as z, it follows that

$$R = R_e + z, \tag{4.26}$$

where R_e is the average radius of the earth. If we substitute eq. (4.26) into eq. (4.25), we find that the weight can now be written as:

$$W = \frac{Gm_e m}{(R_e+z)^2} = \frac{Gm_e m}{R_e^2}\left(1+\frac{z}{R_e}\right)^{-2}. \tag{4.27}$$

The collection of terms involving the earth's properties and the universal gravitational constant are normally expressed in the gravitational constant, g:

$$g \equiv \frac{Gm_e}{R_e^2}, \tag{4.28}$$

so that the weight at height z above the earth's surface is expressed in the form

$$W = mg\left(1+\frac{z}{R_e}\right)^{-2}. \tag{4.29}$$

Equation (4.29) looks strange at first glance. We are accustomed to $W = mg$, so the presence of the dependence on z is unfamiliar. On the other hand, the function of z looks very much like the binomial expansion (4.24). We can assume that $z \ll R_e$, but what does that mean? If we ignore

the dependence on z altogether, then we obtain a very familiar result, that is, $W \cong mg$. If we expand eq. (4.29) in the manner of eq. (4.24) and keep only the first two terms in that expansion, we find that

$$W \cong mg\left(1 - \frac{2z}{R_e}\right). \tag{4.30}$$

This clearly indicates a dependence of weight on height that we do not ordinarily experience. On the other hand, it at least raises the questions, "When does the dependence on height become a significant factor on weight?" and "When does a mass become truly weightless?". The first question can be answered by some straightforward calculations (see Problems 4.9 and 4.10), while the second deserves a bit of discussion. For a body to be weightless, the truncated binomial expansion (4.30) suggests that it would have to be weighed at an altitude $z = R_e/2$. This altitude is sufficiently large that it violates the assumption made in this binomial expansion, that is, $z \ll R_e$. If we look at the exact result (4.29), we see that the body only becomes truly weightless when $z \to \infty$, which is a very different result!

In fact, when the altitude or distance becomes so large that $z \gg R_e$, we would rewrite eq. (4.29) in the form

$$W = mg\left(\frac{R_e}{z}\right)^2\left(1 + \frac{R_e}{z}\right)^{-2}. \tag{4.31}$$

Equation (4.31) can be expanded and truncated as:

$$W \cong mg\left(\frac{R_e}{z}\right)^2\left(1 - \frac{2R_e}{z}\right) \cong mg\left(\frac{R_e}{z}\right)^2. \tag{4.32}$$

The expansion (4.32) clearly indicates that, within a strictly Newtonian world, a body becomes truly weightless only at heights or distances from the earth's surface that are infinitely larger than the radius of the earth. No doubt there are distances for which the weight is significantly less and for which there are practical applications. But, for our purposes, the main point is that the same function can be expanded into different binomial expansions, depending on what information we are seeking. Also, in either instance, we are defining large and small as always, with respect to another dimension or distance. That is, we never say, "z is small" or "z is large." Instead we say that $z \ll R_e$ or that $z \gg R_e$, or, in words, "z is small compared to R_e" or "z is large compared to R_e."

Problem 4.1. Show that eq. (4.7) can be obtained by substituting ix for x in eq. (4.6).

Problem 4.2. Determine the first four terms of the Taylor expansions of $\tan x$ and $\cot x$ about $x = 0$.

Problem 4.3. Determine the first four terms of the Taylor expansions of $\tanh x$ and $\coth x$ about $x = 0$.

Problem 4.4. Use dimensional analysis to determine how the catenary parameter, c, is related to the constant horizontal component of the cable tension, T_0, and its weight per unit length (or unit weight), γ.

Problem 4.5. How much tape sag is permissible to measure a 50 m distance accurately to within 5% ? Within 2% ?

Problem 4.6. What does a body that weighs 10 N at the earth's surface weigh at a height of 10 m? At the peak of Mt. Everest? (*Hint*: You might have to look up some facts about our planet!)

Problem 4.7. According to eq. (4.30), at what altitude would the weight of 10 N at the earth's surface drop to 9 N? To 5 N?

Problem 4.8. Compare the results obtained in Problem 4.7 with more exact results obtained by using eq. (4.29).

Problem 4.9. What does a body that weighs 10 N at the earth's surface weigh on the surface of the moon? On the surface of the planet Pluto? On the surface of the planet Mars? (*Hint*: You might have to look up some facts about our planet's environment!)

Problem 4.10. If the gravitational potential corresponding to Newton's law of gravitation (eq. (4.25)) is given by

$$V_g = -\frac{Gm_e m}{R},$$

find the exact expression that defines this potential as a function of altitude, z, from the earth's surface.

Problem 4.11. Write a binomial expansion of the results of Problem 4.10 to determine the potential energy above the earth's surface to the first order in z.

Problem 4.12. Fill in the missing elements of the following table to two-term order.

Function	Approximation
$\sin x$	
$\cos x$	
$1 - \sin x$	
$1 - \cos x$	

Problem 4.13. Fill in the missing elements of the following table to two-term order.

Function	Approximation
$\sinh x$	
$\cosh x$	
$1 - \sinh x$	
$1 - \cosh x$	

4.2 Algebraic Approximations

As we have seen in Section 4.1, we often drop terms that are of higher order in Taylor series expansions because they will not affect the final answer very much, that is, neglecting those terms does not introduce unacceptable error. We will now look very briefly at some elementary equations of thermal expansion so we can illustrate how we might more generally drop analytical terms to simplify calculations.

When we heat a solid body, the average distance between that solid's atoms increases. Consequently, the linear dimensions of that body—that is, its length, width, or its height—also increase. Thus, assuming that any of the solid's three dimensions is originally of length, L_0, upon heating that produces a temperature difference, ΔT, that dimension increases to the length $L_0 + \Delta L$, where the change in length, ΔL, is given by:

$$\Delta L = \alpha L_0 \Delta T. \tag{4.33}$$

Equation (4.33) tells us that the change in length of a linear dimension is directly proportional to the temperature increase, and that the constant of proportionality is the coefficient of thermal expansion, α. We can rewrite eq. (4.33) as an expression for the heated length of the dimension, L:

$$L = L_0(1 + \alpha \Delta T). \tag{4.34}$$

Suppose the solid we are considering is a sheet of material originally of length L_0 and width W_0. After heating, these two dimensions would each expand according to eq. (4.34) and the plate's original area $A_0 = L_0 W_0$ would expand to the area A:

$$\begin{aligned} A &= L_0(1 + \alpha \Delta T) W_0(1 + \alpha \Delta T) = A_0(1 + \alpha \Delta T)^2 \\ &= A_0 \left[1 + 2\alpha \Delta T + (\alpha \Delta T)^2\right]. \end{aligned} \tag{4.35}$$

Table 4.1 Coefficients of thermal expansion, α, for several common materials.

Material	$\alpha[(°C)^{-1}$, per °C]
Aluminum	24×10^{-6}
Brass	20×10^{-6}
Copper	14×10^{-6}
Glass	$4 - 9 \times 10^{-6}$
Steel	12×10^{-6}
Zinc	26×10^{-6}

The question then arises: Do we need to keep (and use) all three terms in eq. (4.35) to calculate the area change due to heating or cooling?

The answer to the foregoing question depends in part on the coefficient of thermal expansion, α, which is typically a very small number, as can be seen in Table 4.1. Thus, it is tempting to say that because α is small we can neglect the quadratic term in eq. (4.35). And while this may, in fact, be practically alright, in principle it would be wrong, for two reasons. First, if the temperature difference ΔT is large enough, the product $\alpha \Delta T$ might not be negligible. Second, we have cautioned that comparisons should always be made to some reference, so we normally say that it is some dimensional ratio that is small, as we did for $l/2c$ for the catenary. This means that we are making a straightforward numerical estimate. For the present case, the comparable—and proper—statement is that the product $\alpha \Delta T$ is small, so that we can approximate eq. (4.35) as:

$$A \cong A_0(1 + 2\alpha \Delta T). \tag{4.36}$$

From this truncation we can define a surface coefficient of expansion,

$$\gamma \cong 2\alpha, \tag{4.37}$$

where γ is thus derived from our approximating $(1 + \alpha \Delta T)^2$ by $(1 + 2\alpha \Delta T)$.

Problem 4.14. Develop a *volume* coefficient of expansion, β, for a solid of length L_0, width W_0, and height H_0, that parallels the surface coefficient, γ, of eq. (4.37).

Problem 4.15. To what temperature difference would an aluminum solid have to be subjected for the surface coefficient of

expansion to produce errors of 1% in the area change compared to the exact area change?

Problem 4.16. To what temperature difference would an aluminum solid have to be subjected for the volume coefficient of expansion to produce errors of 1% in the area change compared to the exact area change?

4.3 Numerical Approximations: Significant Figures

We now shift our attention to approximations that we make both in measurements and in calculations, that is, we turn to the subject of significant figures. All measurements and virtually all calculations involve approximations or truncations and, therefore, they involve error. When measuring things we try to minimize these errors by being very careful about what we read and record. Although analog displays have been almost completely displaced by digital displays, it is worth revisiting the "good old days" to emphasize an important point about what we regard as significant.

In Figure 4.3 we show an old-fashioned analog display with a graduated scale that goes from 0 to 50 V. The needle points to a number between 12 and 14, and since there are no lines or gradations between 12 and 14, we have to estimate where the needle points within that 2 V interval. Since the needle appears to be about 20% of the distance between these numbers, we estimate that the added voltage measured is $0.20 \times (14 - 12) \cong 0.40$ V, so that the correct reading is 12.4 V. We would characterize this reading as "good to *three significant figures*" because two digits are read directly from the graduated scale, and the third digit is estimated.

It is important to recognize that the number of significant figures is *not* determined by the placement of the decimal point. Had the voltage scale been from 0 to 5 V on the meter in Figure 4.3, we would have recorded a voltage of 1.24 V good to the same three significant figures because we would have directly read 1.2 V plus 20% of the distance between 1.2 and 1.4 V.

We show some examples of how numbers are written in Table 4.2, together with assessments of the number of significant figures of each. The confusion arises because of the presence of terminal zeros. In general, we don't know whether those zeroes are intended to signify something, or whether they are placeholders to fill out some arbitrary number of digits.

It is equally important to recognize that a very similar situation is confronted when doing calculations. Much of the data that engineers and

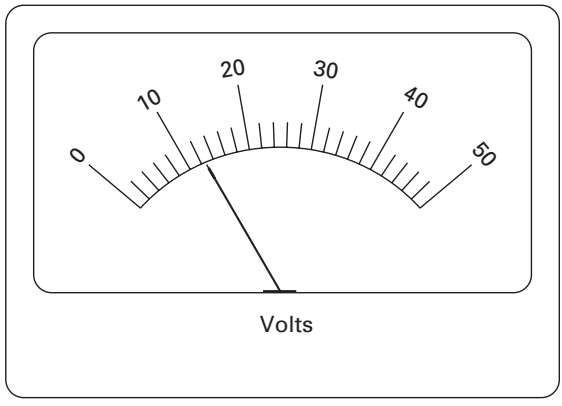

Figure 4.3 An old fashioned analog meter, the standard face before the advent of the digital display. We still see them in many automobile instrument panels, and there are some people who still wear analog watches—but these are uses in which accuracy beyond the gradations is seldom critical. When measuring in the lab and interpreting the results, however, it becomes quite important to know just how many significant figures should be recorded.

Table 4.2 Examples of the ways numbers are typically written and assessments of the number of significant figures that can be assumed or inferred. Confusion arises because of the unstated meaning of the terminal zeroes.

Measurement	Assessment	Significant Figures
9415	Clear	Four
9400	Possibly Confusing	Two (94×10^2) or three (940×10^1) or four (9400)
52.0	Clear	Three
63.2	Clear	Three
6.32	Clear	Three
0.00632	Clear	Three
6.32×10^5	Clear	Three
0.041	Clear	Two
0.0410	Possibly Confusing	Two (0.41) or three (0.0410)
0.00008	Clear	One

scientists use is given only to a limited number of significant figures, sometimes as few as one. For example, there is a much-used material parameter called the *modulus of elasticity*. Denoted by E, this modulus is 30×10^6 lbf/in^2 in British units, implying that at most only two figures (i.e., 30) are significant. It is possible to infer that there is only one significant figure here, but in that case we should write $E = 3 \times 10^7$ lbf/in^2.

Much of the confusion could be mitigated or even eliminated if all technical calculations and experimental data were written in *scientific notation*, wherein numbers are written as products of another number and a power of 10, and where the "new" number is normally between 1 and 10. Thus, numbers both large and small can be written in one of two equivalent, yet unambiguous forms:

$$256{,}000{,}000 = 2.56 \times 10^8 = 0.256 \times 10^9,$$
$$0.000075 = 7.5 \times 10^{-5} = 0.75 \times 10^{-4}.$$

In scientific notation, the number of *significant figures* is equal to the number of digits counted starting from the first nonzero digit on the *left* to either (a) the last nonzero digit on the right if there is no decimal point, or (b) the last digit (zero or nonzero) on the right when there is a decimal point. This notation or convention assumes that terminal zeroes without decimal points to the right signify only the magnitude or power of ten.

We should always remember that *we cannot generate more significant digits or numbers than the smallest number of significant digits in any of our starting data.* In other words, the results of any calculation or measurement are only as accurate as the least accurate starting value. We illustrate this with three examples of multiplication and division showing that the number of significant figures in the result is equal to the smallest number of significant figures in any of the calculation's components:

$$21.982 \times 3.72 = 81.77304 \rightarrow 81.8,$$
$$101.572 \times 0.0031 = 0.3147337 \rightarrow 0.31,$$
$$789.30 \div 0.05 = 15{,}786 \rightarrow 2 \times 10^4.$$

It is far too easy to become captivated by all of the digits that pop up in the displays of our electronic calculators or in computer printouts, but it is really important to remember that *any calculation is only as accurate as the least accurate value we started with.*

In addition and subtraction, the same principle applies. Thus, here we compare the positions of the last significant figure of each number relative to its decimal point because the one that is furthest to the left defines the position of the last allowable significant figure of the sum or difference

being calculated. For example,

$$\begin{array}{r} 53.24 \\ +3.333 \\ +2.4 \\ \hline 58.9 \end{array} \qquad \begin{array}{r} 489.3213 \\ -5.487 \\ \hline 483.834 \end{array}$$

Another important issue when dealing with numbers is that of *round off*, that is, when should we round off numbers, or should we round them off at all? We generally round off numbers at the end of a calculation because dropping insignificant numbers earlier increases uncertainty. The standard convention for rounding off uses the number 5 as its benchmark: Numbers less than 5 following the last retained significant digits are dropped, while numbers greater than 5 cause us to add 1 to the last significant digit retained. If the digit to be rounded or dropped is itself a five, we make the preceding digit even (i.e., even digits are left so, while odd digits are "rounded up" to the next even digit). Thus, for example,

$$5.017 \rightarrow 5.02,$$
$$5.015 \rightarrow 5.02,$$
$$5.014 \rightarrow 5.01,$$
$$5.025 \rightarrow 5.02.$$

These results also indicate the degree of uncertainty in the true value of a number that has been rounded off. From the data just given and the rules behind it, we see that the number 5.02 could mean a number that is actually between 5.015 and 5.025.

Finally, it is worth noting that there are numbers that have unlimited significant figures. Some are whole numbers representing an exact count, and thus contain an unlimited number of significant figures. They are usually written without any digits after the decimal point, or they may not have a decimal point at all. To indicate such a number, we might write "35." or, as in the formula for the circumference of a circle, $C = 2\pi r$, wherein the number "2" represents an exact count and is written without a decimal point. The number π is itself a number that has an infinite number of significant figures, as does e, the base of Naperian logarithms. However, we write "35.0" or "2.0" when we want to indicate that we are measuring something to the first decimal place.

Whether reporting measurement data or doing calculations, we should always keep in mind the significance of our initial data so that we can assess the validity of our results.

Problem 4.17. Round off each of the following numbers to two (2) significant figures:
(a) 5.237 (b) 0.82549 (c) 81.356 (d) π
(e) 6.2305 (f) 0.0428 (g) 10.45 (h) 4.035

Problem 4.18. Round off each of the following numbers to three (3) significant figures:
(a) 5.237 (b) 0.82549 (c) 81.356 (d) π
(e) 6.2305 (f) 0.0428 (g) 10.45 (h) 4.035

Problem 4.19. Complete the following multiplications and express the results to the correct number of significant figures:
(a) $(6.28 \times 10^3) \times 2.712$ (b) 43.32×0.3
(c) 928×4.23

Problem 4.20. Do 99.9 and 100.1 have the same number of significant figures? Explain your answer.

Problem 4.21. Estimate the ranges within which each of the following numbers lie:
(a) 7.7 (b) 7.70 (c) 1200 (d) 1.200×10^{-3}

4.4 Validating the Model–I: How Do We Know the Model Is OK?

There are two issues that arise when we speak of the validity or correctness of a model. The more obvious one is whether or not the model can predict the measured or observed behavior of whatever object or device is being modeled. Thus, if we are modeling the period of the oscillations of a pendulum, as we started to do in Chapter 2, we could reasonably expect that changes in the pendulum length would produce oscillations at correspondingly different periods or frequencies. As we see from eq. (2.2), if we double the length l of a pendulum, we would expect its period to increase by about 41% . Similarly, were we doing pendulum experiments on the moon, we would expect to see an increase in the period of about 145%. These predictions of the pendulum's behavior are confirmed by the available experimental data, and so the model is validated. Alternatively, given empirical data without an underlying theory, we could construct a model to explain the empirical data—although it is also quite likely that the (new) model or theory would be further tested by making predictions about experiments as yet undone or measurements as yet untaken.

(We note parenthetically that the measurement [and containment] of experimental error is a complex subject that is closely linked to the field or discipline in which the experiment is intellectually housed. However, there

are some fundamental ideas about error and about statistics that apply generally, and we will introduce them in Sections 4.5–4.8.)

The less obvious question about model validity is concerned with the inherent consistency and validity of the model. If we hark back to the modeling meta-principles outlined in Section 1.2, we see issues and questions that pertain directly to model validation. For example, have we identified the right governing principles? Have we used the right equations? And, is the model consistent with its principles and assumptions? The first two of these questions are about ensuring that we apply the proper principles and formulations when we try to find what we are seeking. Again, when modeling the pendulum, our basic principles are Newton's law of motion, and our assumptions will depend on whether we are anticipating small angles of oscillation or large. As we will see in Chapter 7, a linear equation of motion suffices in the former case, while a complete nonlinear formulation is needed for the latter (large oscillations).

4.4.1 Checking Dimensions and Units

There are several checks or tests we can bring into play while we build models and approximate the mathematics. The first is the application of the principle of dimensional homogeneity (cf. Section 2.2), which requires that each term in an equation has the same net dimensions. For example, the *stiffness* or spring constant of a cantilever beam, k, can be written in terms of the beam's length, L, second moment of its cross-sectional area (commonly but erroneously called the "moment of inertia"), I, and modulus, E, as:

$$k = \frac{3EI}{L^3}. \tag{4.38}$$

The physical dimensions of the parameters in eq. (4.38) are $\mathsf{F/L}$ for the spring constant, L for the beam length, L^4 for I, and $\mathsf{F/L}^2$ for the modulus. Thus, we can apply the principle of dimensional homogeneity to ensure that eq. (4.38) has the correct dimensions and is dimensionally consistent:

$$[k] = (\mathsf{F/L}) = \left[\frac{3EI}{L^3}\right] = \frac{1 \times (\mathsf{F/L}^2) \times \mathsf{L}^4}{\mathsf{L}^3} = (\mathsf{F/L}). \tag{4.39}$$

If the dimensions of all the terms in an equation or model are not known, as is sometimes the case, then the principle of dimensional homogeneity can be applied to properly determine the dimensions of the unknown quantity. In the case of the cantilever beam, if we didn't know the dimensions of I, we would solve eq. (4.38) for I and then apply the principle of

dimensional homogeneity again:

$$[I] = \left[\frac{kL^3}{3E}\right] = \frac{(\mathsf{F}/\mathsf{L})\mathsf{L}^3}{\mathsf{F}/\mathsf{L}^2} = \mathsf{L}^4. \tag{4.40}$$

We can also take the principle of dimensional homogeneity one step further and use it as a guiding principle for checking the specific *units* used in a numerical calculation. If we measured the properties of a particular cantilever beam, say a standard (12 in) steel ruler to be used in a classroom project, we would find

$$\begin{aligned} E &= 2.05 \times 10^2\,\text{GPa}, \\ I &= 6.78 \times 10^{-5}\,\text{cm}^4, \\ L &= 2.81 \times 10^{-1}\,\text{m}. \end{aligned} \tag{4.41}$$

If we substitute these values into eq. (4.38), we see immediately that we have a mismatch of units:

$$k = \frac{3(2.05 \times 10^2\,\text{GPa})(6.78 \times 10^{-5}\,\text{cm}^4)}{(2.81 \times 10^{-1}\,\text{m})^3}. \tag{4.42}$$

The units' mismatch is easily rectified if we use proper unit conversions, that is,

$$k = \frac{3\left[2.05 \times 10^2 \times 10^9\ \text{Pa}\left(\frac{\text{N/m}^2}{\text{Pa}}\right)\right]\left[6.78 \times 10^{-5}\ \text{cm}^4\left(\frac{\text{m}}{10^2\text{cm}}\right)^4\right]}{(2.81 \times 10^{-1}\ \text{m})^3}, \tag{4.43}$$

or

$$k = \frac{3\left[2.05 \times 10^{11}\ \text{N/m}^2\right]\left[6.78 \times 10^{-13}\ \text{m}^4\right]}{(2.81 \times 10^{-1}\ \text{m})^3} = 1.88 \times 10^1\ \text{N/m}. \tag{4.44}$$

Two final notes here. First, it is generally a better strategy to write all of the data to be used in the same system of units at the beginning of a calculation as this reduces the chance for error. Thus, here we could have converted the units immediately after the measurements were taken. Second, note that we have used scientific notation in both writing the measurements and performing the arithmetic. Thus, there can be no doubt about the number of significant figures in the answer (4.44).

4.4.2 Checking Qualitative and Limit Behavior

Model validation is integral to the modeling process. Models are validated by having their predictions confirmed experimentally, or statistically, or by some other quantitative means. In both our physical and mathematical reasoning we must make explicit our assumptions and their limits, and we must ensure that our mathematics does indeed reflect the physics we are modeling. In addition to looking at numbers, the mathematical behavior should "feel right" in qualitative terms. We did just such qualitative analysis at the beginning of this section when we described the expected behavior of the pendulum as a function of its length, l. Similarly, as also indicated by eq. (2.2), it feels intuitively right that pendulums will swing faster and have shorter periods in stronger gravitational fields. Thus, when we are constructing mathematical models, and especially when we are making mathematical approximations, we need to take care that we are admitting mathematical behaviors that are qualitatively appropriate.

Still another example of such reasoning is available from our just-completed dimensional check of the stiffness of a beam. Here we rewrite eq. (4.38) in a form that explicitly identifies the physical meaning of each parameter that appears in the equation:

$$(k = \text{beam stiffness}) \propto \frac{(E = \text{material stiffness})(I = \text{cross} - \text{sectional 2nd moment})}{(L = \text{beam length})^3}. \tag{4.45}$$

Equation (4.45) can be viewed through the eyes of a structural engineer talking about the meaning of its mathematical version, eq. (4.38). It supports the engineer's intuitions as follows. It stands to reason that the beam's stiffness is proportional to the material stiffness, that is, it increases or decreases as does E. The beam's stiffness is also proportional to the second moment of the beam's cross-section, I. It also is intuitively pleasing that the stiffness is inversely dependent on the length, so that the beam's stiffness increases as L becomes very small and decreases as L becomes very large. Finally, if we look at the limiting cases of each parameter decreasing to zero or becoming indefinitely large, we would see that each of the trends exhibited by eq. (4.45) is consistent with the reasoning just outlined, as well as with our practical experience of beams in the real world.

Reasoning about the way that variables appear in equations is of second nature in mathematical modeling, and we will have many opportunities to invoke such reasoning in the discussions of applications that follow. One simple example is afforded by the fundamental frequency of free vibration of a cantilever beam, ω, of mass density, ρ, and cross-sectional area, A,

with a mass, m, at its tip. That frequency is, approximately,

$$\omega \cong \sqrt{\frac{3EI/L^3}{\rho AL(1 + m/\rho AL)}}. \tag{4.46}$$

Does eq. (4.46) exhibit the right qualitative and limit behavior? It does. It reduces to a well-known result for a cantilever beam when the tip mass, m, vanishes, and eq. (4.46) correctly describes the frequency of a mass-less beam with a tip at its end when that tip mass gets so large that it dominates the beam mass.

It may seem that much of what has been said in this section is *common sense*. It is, as long as it is commonly applied! To invert a popular saying, "If we expect our model to be a duck, then it should look like a duck, walk like a duck, and quack like a duck."

Problem 4.22. By what percentage would the period of a pendulum change if its length was halved? If it was reduced by one-third? If the length was reduced to one-third of its original length?

Problem 4.23. Explain why the pendulum period increases by 145% on the moon.

Problem 4.24. How would the period of a pendulum change, compared to its value on earth, if the pendulum was on Mars? On Pluto?

Problem 4.25. How would the period of a pendulum change as a function of its height, h, above the surface of the earth? (*Hint*: The variation of the gravitational acceleration g can be represented as a function of h from Newton's law of gravitational attraction.)

4.5 Validating the Model–II: How Large Are the Errors?

Building mathematical models means using numbers derived from experimental or empirical data, or from analytical or computer-based calculations. Errors are thus always present, whether due to data reading or data manipulation. Since error is always present, we turn now to a discussion of error and statistics—the way we deal with error.

4.5.1 Error

Error is defined as the difference between a measured (or calculated) value and its true or exact value. Error is *always* present. How much error is present depends on how skillfully the data is read or manipulated. Therefore, error analysis should be a part of every modeling process.

There are two types of error. *Systematic error* occurs whenever an observed or calculated value deviates from the true value in a consistent way. Systematic error occurs in experiments when instruments are improperly calibrated because their output varies during use. Thus, instruments must be properly calibrated before an experiment is run and before data is measured and recorded. Improper calibration affects both analog and digital data recorders, although analog displays are also subject to other kinds of systematic error, such as a bent needle on a meter face such as that shown in Figure 4.3. Systematic error also affects calculations, although this is more controllable as it is likely due to using incorrect values of "known" variables or to improper control of the number of significant figures retained during the calculation process.

Random errors are, not surprisingly, due to chance. They arise largely in experimental work because unpredictable things happen and because not everything in an experimental set-up is known with complete certainty: Connections can be loose or break altogether, dirt may get into a sensitive moving part, or the amount of friction present in a moving part may not be controllable. The resulting random error varies in both magnitude and sign. The laws of statistics help us to describe and account for the distribution of such random errors. Indeed, it has been said that randomness is a mathematical model for variability that cannot be explained in a deterministic way.

The *absolute error* is defined as the difference between the true or expected value, X_e, and the measured value, X_m, that is, as $X_e - X_m$. The true value, X_e, may be known or it may have an expected value based on a calculation or some other data source. The *relative error* is the absolute error divided by the measured value, that is, $(X_e - X_m)/X_m$.

The statistic found most useful is the *percentage error*, which is the percentage-based relative error:

$$\% \text{ error} = (100)\frac{(X_e - X_m)}{X_m}. \tag{4.47}$$

For example, suppose that an ammeter has a systematic error of +2 A (amperes) because of either a bent needle (analog) or improper calibration

(digital or analog). When the display reads 100 A the percentage error is

$$\% \text{ error} = (100)\frac{(102 - 100)}{100} = 2\%,$$

while if the same ammeter reads 20 A the percentage error is

$$\% \text{ error} = (100)\frac{(22 - 20)}{20} = 10\%.$$

The percentage error is much larger in this instance, providing another example of how scale affects results!

Similarly, errors are introduced when series expansions are truncated (cf. Section 4.1.2). For example, for $\theta = \pi/12(15^\circ)$, the percentage error incurred by replacing $\sin x$ with x is:

$$\% \text{ error} = (100)\frac{(\sin \pi/12 - \pi/12)}{\pi/12} = -1.14\%.$$

Note that *errors* and *mistakes* are not the same thing. Errors are defined as the difference between a true or expected value and a measured (or calculated) value. Further, as we discussed above, some error is unavoidable. On the other hand, *mistakes* are blunders made by the person doing the experiment (or analysis or calculation). Blunders are made by reading or recording erroneous data, using instruments inappropriately (e.g., improperly calibrated instruments, inadequately sensitive meters), using the wrong formulas, using inconsistent or wrong units, and so on. These kinds of mistakes can—and obviously should—be avoided.

4.5.2 Accuracy and Precision

Since we have to contend with systematic and random errors, as well as with the hopefully rare mistake, it is important that we be able to estimate the effects of these errors and mistakes.

Accuracy is defined as a representation of how close a measured or calculated value is to an established or true value. In experimental work, accuracy is usually expressed as a percentage of the maximum scale value. Thus, voltages read on a 100 V scale with an accuracy of 5% are accurate to within ± 5 V.

Precision is defined in terms of the ability to reproduce a set of data with a specified accuracy. The more precise a set of readings or calculations, the closer the individual readings or calculations are *to each other*. Thus, suppose we measured an input voltage that is known to be 50 V with the voltmeter having an accuracy of 5%. Five individual readings are taken and recorded as, respectively, 54, 53, 55, 53, and 55 V. These clearly fall

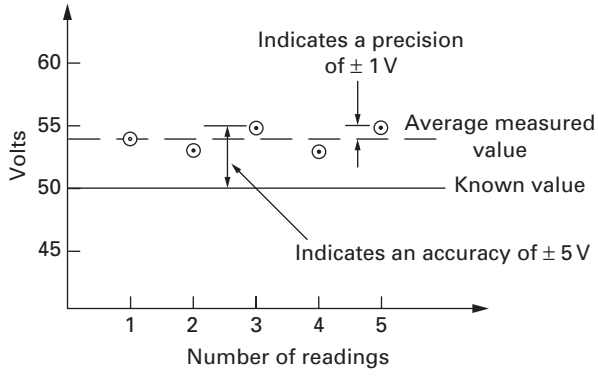

Figure 4.4 Some (made-up) experimental data that illustrates: *accuracy*, the closeness of the measured value to an established value, and *precision*, the ability to reproduce a set of measurements within a specified accuracy. These data reflect measurements that are rather precise, yet relatively inaccurate.

within the meter's accuracy bounds of ± 5 V. Since the average or mean reading of the five readings is 54 V, and since the maximum deviation from this mean of any one of the measurements is 1 V, the precision of the five measurements is determined to be $\pm 1\%$ (remember that the meter has a 100 V scale). As we illustrate in Figure 4.4, our little virtual experiment has produced precise but relatively inaccurate readings.

It is worth noting that the accuracy of a measuring device is controlled by its sensitivity because it is the *sensitivity* that identifies the minimum amount of change that the device can detect and indicate. Suppose we wanted to measure very small voltages, say less than 1 millivolt (mV). Our trusty voltmeter allows us to choose one of three measurement ranges: 0–50 V, 0–2.5 V, or 0–5 mV. With either of the first two ranges we will see no reading at all. However, with the third scale, 0–5 mV, there will be a noticeable measurement that can be recorded. Thus, moving from either of the first two scales to the third produces a more sensitive voltmeter, and so our readings will be more accurate. Hence, we see how scale influences sensitivity and, therefore, accuracy.

Problem 4.26. Draw two circular archery targets and use them to depict the "hit" patterns of (a) an archer who is accurate, but not precise; and (b) an archer who is precise, but not accurate.

4.6 Fitting Curves to Data

Graphical presentations of calculations and experimental results are the most convenient—and often the most informative—presentation of data available. We can spot trends, identify discontinuities, and generally get an intuitive "feel" for what the data "says" when we look at plots or curves. Given this very human proclivity, how do we draw curves for a given collection of points? That is, since plotted data points rarely align themselves perfectly on a known or identifiable curve, how do we fit a curve through them? Still further, how do we generate the "best fit" of a curve through the data?

The short answer to these questions is in a familiar spirit: It depends on what you want. If the accuracy of the curve is not too important, and if we're only looking for a rough, qualitative idea of how one variable depends on another, then we can draw the curve "by eye." That is, we draw a smooth curve that seems to go through the plotted data points with an eye to perhaps "distributing" the data in roughly equal amounts above and below the curve drawn, as we have done in Figure 4.5.

Often, greater accuracy is desirable, as when we want to *interpolate* to obtain values between measured values, or even more so when we want to *extrapolate* to estimate values beyond the range of the measured values. Extrapolation can easily magnify errors in the estimated values, so that greater accuracy is quite important. Further, extrapolation is most accurate when the curve drawn is a straight line.

The *method of least squares* is the most commonly used approach to obtaining a best straight line through a series of points. It assumes that all of the *scatter*, the variation of the data from the drawn curve, derives

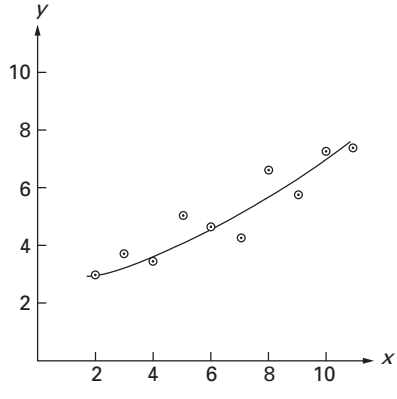

Figure 4.5 A best-fit curve that is drawn by hand using visual estimation (i.e., "drawn by eye").

from error in measuring one of the variables. That variable is chosen as the ordinate for the axes on which the straight line will be plotted. Then the best-fit straight line is the one that has the minimum errors in the ordinate.

We are thus looking for an equation of the usual form

$$y = mx + b, \tag{4.48}$$

where b is the y-intercept with [b] = [y], and m is the slope with [m] = [y/x]. We first define the error in each reading as the difference in the ordinate between the measured value, y_i, and the straight line's ordinate, $(mx_i + b)$, for all values of the abscissa, x_i:

$$E_{y_i} = y_i - (mx_i + b). \tag{4.49}$$

We define a measure S of the total error as the sum of the square of the errors at every point on the abscissa, x_i, where values of the ordinate, y_i, are given, that is, as

$$S = \sum_{i=1}^{n} (E_{y_i})^2 = \sum_{i=1}^{n} [y_i - (mx_i + b)]^2. \tag{4.50}$$

The minimum of the measure of the total error is then found by differentiating S with respect to m and b and so determining the values of m and b needed to plot eq. (4.48):

$$\begin{aligned} \frac{\partial S}{\partial m} &= 2 \sum_{i=1}^{n} [(y_i - mx_i - b)(-x_i)] \\ &= -2 \sum_{i=1}^{n} x_i y_i + 2m \sum_{i=1}^{n} x_i^2 + 2b \sum_{i=1}^{n} x_i = 0, \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} \frac{\partial S}{\partial b} &= 2 \sum_{i=1}^{n} [(y_i - mx_i - b)(-1)] \\ &= -2 \sum_{i=1}^{n} y_i + 2m \sum_{i=1}^{n} x_i + 2nb = 0. \end{aligned} \tag{4.52}$$

Equations (4.51) and (4.52) are a pair of linear algebraic equation that can be solved (see Problem 4.28) to yield the following values of m and b:

$$m = \frac{n \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right)}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2}, \tag{4.53}$$

and

$$b = \frac{\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}. \tag{4.54}$$

Note that eqs. (4.53) and (4.54) have different physical dimensions that depend on the particular physical problem being modeled (see Problem 4.28).

Consider now the data displayed in the first two columns of Table 4.3, which are the result of another, virtual experiment. We will now determine the best straight line that can be drawn through the data. First, we calculate the products shown in the third and fourth columns of Table 4.3. Then we sum all four columns to find the data in the last row of the table, which are then substituted into eqs. (4.53) and (4.54) to find $m = 0.85$ and $b = 1.26$. The best straight-line fit through the data of Table 4.3 is, then,

$$y = 0.85x + 1.26. \tag{4.55}$$

Equation (4.55) is plotted in Figure 4.6, together with the data from Table 4.3, and we see that the straight line seems to fit the data pretty well. Can we characterize the *quality* of that fit, that is, just how well does eq. (4.55) fit the given data? The quality of fit is expressed in terms of R^2, called "R squared," which describes how well a curve *regresses* toward the

Table 4.3 A table of data from a virtual experiment used to calculate the best-fit straight line approximation shown in Figure 4.6.

i	x_i	y_i	$x_i y_i$	x_i^2
1	0	1.0	0	0
2	1.0	2.1	2.1	1.0
3	2.0	2.8	5.6	4.0
4	3.0	3.6	10.8	9.0
5	4.0	5.0	20.0	16.0
6	5.0	5.5	27.5	25.0
7	6.0	8.0	48.0	36.0
8	7.0	6.4	44.8	49.0
9	8.0	7.4	59.2	64.0
	$\sum_{i=1}^{9} x_i = 36.0$	$\sum_{i=1}^{9} y_i = 41.8$	$\sum_{i=1}^{9} x_i y_i = 218.0$	$\sum_{i=1}^{9} x_i^2 = 204.0$

data from which it was derived. R^2 is a number between 0, which indicates no fit at all, and 1, which describes a perfect fit. (There are many mathematical and statistical computational packages that include the formulas needed to calculate R^2.)

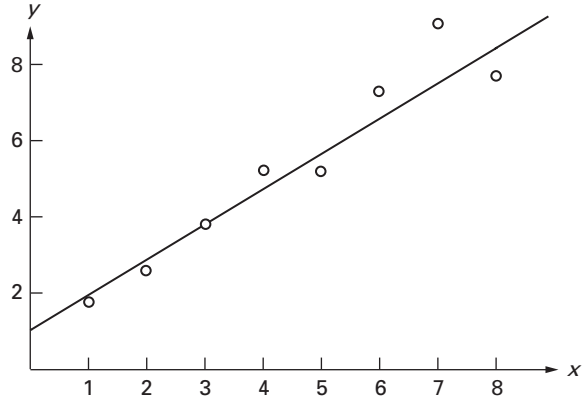

Figure 4.6 A best-fit straight line for the data in Table 4.3 produced by least squares. It is analytically represented as $y = 0.85x + 1.26$.

Problem 4.27. Verify the final forms of eqs. (4.51) and (4.52).

Problem 4.28. Verify the equations for m and b given in eqs. (4.53) and (4.54).

Problem 4.29. Discuss and explain the dimensional differences between eqs. (4.53) and (4.54).

Problem 4.30. Verify the terms in the third and fourth columns of Table 4.3, as well as the sums of all four columns.

Problem 4.31. Verify the calculations of m and b found from the results in Table 4.3.

4.7 Elementary Statistics

What do we do *after* we have recorded a bunch of measurements or calculated several values of something? A more meaningful phrasing of this question would be: How do we organize and present our results so that we are better able to understand and communicate the data? Our answer to this question comes in two parts. In the first, we define the meaning of average, while in the second, we discuss ways of drawing curves through data.

4.7.1 Mean, Median, and Standard Deviation

We often want to average our results when making several measurements or calculating several values of something. There are several ways of defining the meaning of average, but we will limit our discussion to two: the mean and the median.

In Figure 4.4 we showed data from a virtual experiment whose individual measurement readings (and, occasionally, model calculations) vary from one another. We want to deal with a single value, a best estimate of the magnitude of the entire set of readings. We will take the average or mean of a *sample* of n measurements as such a best estimate, where the *arithmetic mean* or *sample mean* $\bar{x}$ is defined as the sum of all of the individual readings x_i divided by the number of readings, n:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{4.56}$$

Note that the calculation of the mean of a set of values given by eq. (4.56) strongly resembles the way that the centroids of areas are calculated, and for good reason!

There is one other measurement that is often cited as a meaningful indicator of an "average" of a number of readings and that is the *median*, which is defined as the measured value that is at the middle of the distribution. The median removes any bias that might be introduced by a few values that differ significantly from the mean. For example, in the virtual voltmeter experiment of Section 4.5.2, the median is 54, which is the same as the mean. On the other hand, had the five readings been 54, 53, 65, 53, and 55 V, then the mean rises to 56 V, while the median stays at 54.

In Table 4.4 we show a collection or sample of 100 noise level measurements of the noise due to traffic as measured in a schoolyard playground. In addition to traffic noise, the microphones also picked up the occasional noise due to children in the playground who, excited by the experiment, made some loud sounds as they passed by. We see that for these measurements the mean is higher than the median, which is likely due to the relatively large number of readings in the 90–91 dB interval.

In addition to identifying the mean as our best estimate, we would like to estimate the spread or dispersion of the set of measurements about the mean. Clearly, if this estimate of the spread is small in some sense, then we can attach a high precision to the mean $\bar{x}$. The accepted statistical measure of this estimate of dispersion is the *sample variance*, s^2, defined in

Table 4.4 A sample of 100 noise level measurements (in decibels (dB)) made in a schoolyard playground.

Decibels		Number of Observations
	90–91	x x x x x x x x
	88–89	x x x x x
	86–87	x x x
	84–85	x x
	82–83	x x x x
	80–81	x x
	78–79	x x x
	76–77	
	74–75	x x x x x x
	72–73	x x x x
Mean	70–71	x x x x x x
	68–69	x x x
Median	66–67	x x x x x x x
	64–65	x x x x x
	62–63	x x x x x x x x x
	60–61	x x x x x x x x x x x x x
	58–59	x x x x x x
	56–57	x x x x x x x x
	54–55	x x x x
	52–53	x
	50–51	

terms of the deviation of each reading from the mean, $(x_i - \bar{x})$, as:

$$\begin{aligned} s^2 &\equiv \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n - 1} \\ &= \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2. \end{aligned} \tag{4.57}$$

The *standard deviation*, s, is defined as the square root of the sample variance:

$$s \equiv \left[\frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right]^{1/2}. \tag{4.58}$$

We often see the symbol σ used for the standard deviation, but that usage is correct only when the calculation is performed for the *total population* or the complete set of all the objects being measured. When we are taking

readings or doing calculations, we are taking a *sample* of all of the values that could, in principle, be obtained. In that case, s is the correct notation for the standard deviation of that sample. (Similarly, when the calculation of a mean is done for an entire population, it is denoted by μ, rather than $\bar{x}$.)

Note that in calculating the standard deviation, the deviation of each value or reading from the mean is squared before being added to the comparable deviations of the rest of the readings. This is done to eliminate the sign differences that occur because the deviation $(x_i - \bar{x})$ can be positive or negative, depending on whether the reading x_i is greater or smaller than the mean $\bar{x}$. Thus, only positive numbers are added when the standard deviation is calculated. Also note that eq. (4.58) clearly suggests that the best way to increase the precision of the answer is to increase the number of readings or calculated values. Indeed, an infinite number of measurements would, in theory, produce perfect precision because the standard deviation vanishes in the limit $n \to \infty$. We also point out that just as the calculation of the mean parallels the calculation of the location of the centroid of an area about one axis, the calculation of the variance (eq. (4.57)) parallels the calculation of the second moment of area about that same axis.

Notwithstanding the physical analogy just given, the interpretation of the standard deviation, s or σ, is difficult because its units are squares of the units of the variable, x. However, we can give meaning to the standard deviation when we relate it to the mean of the data set, $\bar{x}$ or μ. This meaning is embedded in the *Empirical Rule* that tells us, approximately, where the data points lie with respect to the mean. The following heuristics describe the data set that underlies a distribution that is, approximately, a mound shape:

- almost all of the data points lie within 3 standards of deviation of the mean of the data set, that is, within the window $(\bar{x} \pm 3s)$ for samples and within the window $(\mu \pm 3\sigma)$ for complete populations;
- some 95% of the measurements lie within 2 standards of deviation of the mean of the data set, that is, within the window $(\bar{x} \pm 2s)$ for samples and within the window $(\mu \pm 2\sigma)$ for complete populations; and
- some 68% of the measurements lie within 1 standard of deviation of the mean of the data set, that is, within the window $(\bar{x} \pm s)$ for samples and within the window $(\mu \pm \sigma)$ for complete populations.

4.7.2 Histograms

Another way of displaying measured data is the *histogram* or *bar chart* in which a distribution of the frequency of occurrence of the measured quantity is displayed. The histogram's abscissa indicates the values recorded,

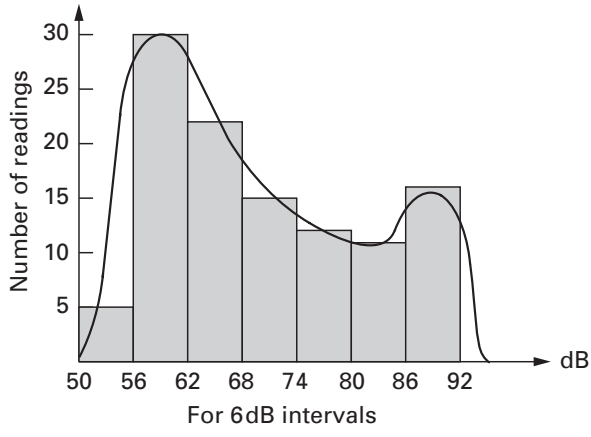

Figure 4.7 A histogram of the noise measurements given in Table 4.4, with a continuous approximation of the same noise level data superposed as a dotted line. Here the data were taken and recorded in 6-dB windows or intervals. For example, there are 30 measurements registered in the 56–62 dB window.

while its ordinate represents the number of times the values occur. The histogram shown in Figure 4.7 displays the same data given in Table 4.4 with the measured sound pressure levels grouped in 6-dB intervals or windows. Thus, the bar between 56 and 62 dB represents the total number of measurements that registered, respectively, 56, 57, 58, 59, 60, or 61 dB. Two questions occur immediately: Why construct histograms? and How big should the intervals be?

The main reason for constructing a histograms is that it offers a graphic depiction of the frequency of events, so that problematic repetitions of particular events are readily identified. Histograms can also be used to generate approximate plots based on the data they express. For example, Figure 4.7 also shows a continuous approximation of its 6-dB histogram. Both the histogram and its continuous counterpart show us that the largest number of readings of outdoor noise in the schoolyard occur in the 56–62 and 86–92 dB windows. This prompts us to inquire about the cause(s) of readings at these two levels. In response, we can identify the peak in the 86–92 dB window as deriving from the children yelling at the microphone, which in turn allows us to note that the playground noise is more generally at levels less than 86 dB, with the remaining peak occurring at the relatively low levels of 56–62 dB.

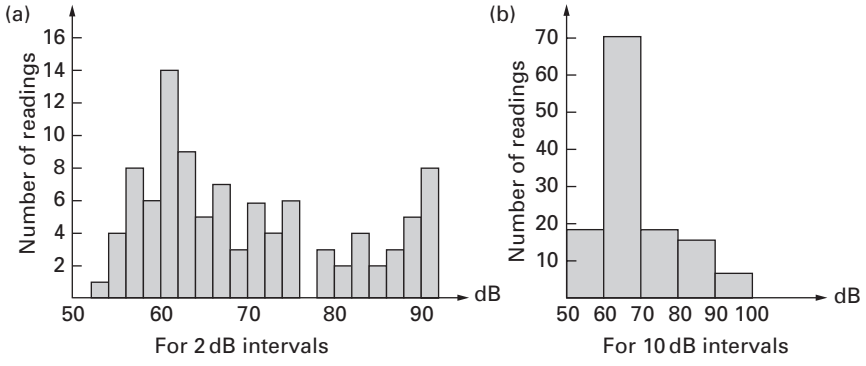

Figure 4.8 Two more histograms of the noise measurements given in Table 4.4. The data were taken and recorded in, respectively, (a) 2-dB and (b) 10-dB windows.

How do we decide on the size of the intervals or windows? We want the interval to be *large enough* to have enough data to minimize the chance of spurious fluctuations, yet *small enough* that we don't throw out data that would indicate serious events within the interval. The data of Table 4.3 (and the 6-dB histogram of Figure 4.7) are displayed in Figure 4.8 in histograms with intervals of (a) 2 dB and (b) 10 dB. We see that with the larger interval we have lost the (identifiable) peak due to the children's screaming, while with the smaller interval we have many more peaks and fluctuations. As a practical matter, experience suggests that the number of bars in a histogram should roughly equal the square root of the number of data entries, $\sqrt{n}$.

How did we draw the curve representing the continuous version of the histogram in Figure 4.7? First, we assumed the validity of the *continuum hypothesis*, which states that such discrete data can be plotted as a continuous curve. Second, we chose the number of intervals to get a relatively smooth and meaningful curve. Just as with the underlying histogram, this meant going back to the original data (i.e., Table 4.4) to choose an interval size large enough to contain a significant number of points, yet not so large that variations within the interval are drowned out. We constructed Table 4.5 to aid in this process of choosing an interval size. Table 4.5 organizes the data in Table 4.4 in terms of the number of points within intervals of length Δ centered around 66 dB: There are 13 readings in the interval of $\Delta = 4$ dB, 27 in the interval of $\Delta = 8$ dB.

A plotted curve of the data of Table 4.5, in Figure 4.9, helps us better visualize and understand the data. If the length of the measuring interval Δ is too small, say < 4 dB, the density fluctuates a lot and is not representative of the complete picture. If the interval Δ is too large, say > 8 dB, the

Table 4.5 An organizing chart of the data in Table 4.4 that allows us to estimate the number of data points in intervals of varying length Δ. This form of the data enables the drawing of the plot shown in Figure 4.9.

Interval length, Δ	1	2	3	4	5	6	7	8	10	20	30	40	50
Interval, $66 \pm \Delta/2$	66.5 65.5	67 65	67.5 64.5	68 64	68.5 63.5	69 63	69.5 62.5	70 62	71 61	76 56	81 51	86 46	91 41
Number of readings in interval	6	9	12	13	17	19	24	27	37	68	78	85	100
Density	6	4.5	4	3.25	3.4	3.17	3.43	3.34	3.7	3.4	2.6	2.38	2.00

density curve is smoothed out to the extent that all of the meaningful variations have disappeared. Thus, an interval such that $4 < \Delta \text{ dB} < 8$ would appropriately approximate the number of readings as a continuous function of the noise level. We have 100 readings here, so $10 = \sqrt{100}$ histogram bars are appropriate for the range 50–90 dB, resulting in the shown width of 4 dB. However, as with other aspects of modeling, the number of histogram bars is to some extent a matter of taste.

We have not offered any criteria to aid in choosing a measuring interval because there are none. The best path is to organize the data as in Table 4.5, use it to plot a curve such as that in Figure 4.9, and then exercise our best judgment as to the size of Δ.

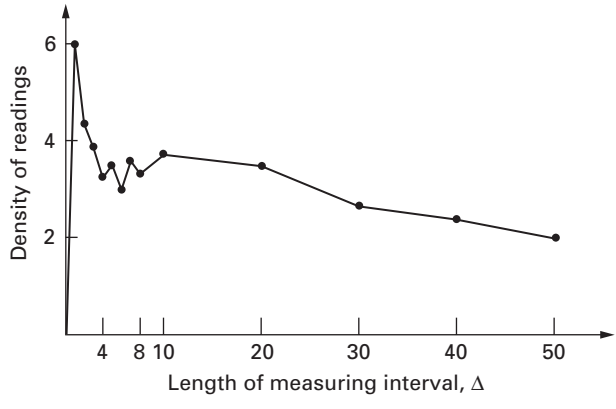

Figure 4.9 An illustration of the *continuum hypothesis,* showing how the density of the readings depends on variation of the measuring interval.

Lastly, all of the calculations outlined in this section can be done on a computer. Care and thought are invaluable because, at best, a computer does only what it's told to do. Given erroneous instructions or bad data, its results will be erroneous and bad!

Problem 4.32. Determine the standard deviation for the data presented in Table 4.4.

Problem 4.33. Draw a histogram for the data in Table 4.4 with 10 intervals of 4 dB width.

Problem 4.34. Show that the square of the sample variance of eq. (4.57) can be cast in the alternative form

$$s^2 = \frac{1}{(n-1)} \left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2 \right].$$

4.8 Summary

We have devoted this chapter to discussions of approximations and their limits, and of model validation, including both qualitative and statistical methods. We have shown the importance of Taylor and algebraic series expansions, including applications to stretched strings (Taylor series of hyperbolic functions), gravitational forces (binomial expansions), and thermal expansion (algebraic approximations). We have emphasized the need to validate models, as well as the roles played by dimensional and qualitative analyses in model validation. We have also stressed the importance of numerical approximations and of significant figures, especially as regards their proper display and interpretation.

Working with mathematical models means that we are constantly using numbers that derive from calculations or experiments. These numbers always incorporate error. We have discussed both random and systematic errors, and how they affect the precision and accuracy of any set of data. We also looked briefly at statistical techniques that could be used to quantify such errors, introducing the concepts of mean, median, and standard deviation. We showed how curve fitting could be used to approximate functions, and we showed illustrative examples using both the least squares method and the continuum hypothesis to develop statistically based numerical approximations.

4.9 Appendix: Elementary Transcendental Functions

The so-called *elementary transcendental functions* are the trigonometric functions ($\sin x, \cos x$), the exponential functions (e^x, a^x), the hyperbolic functions ($\sinh x, \cosh x$), and the logarithmic functions ($\ln x = \log_e x, \log_a x$). We will present some basic results and relationships for these functions, rather than derivations and proofs. Some of the results make use of the notation $i = \sqrt{-1}$, which is central in relating, for example, the trigonometric functions to the exponential. In fact, we will use what famed physicist Richard Feynman called "the most remarkable formula in mathematics":

$$e^{ix} = \cos x + i \sin x. \tag{4A.1}$$

We also note that that the *imaginary* (as it is often called) number i is often denoted instead by $j = \sqrt{-1}$, especially by the electrical engineering community, but we will stick to the traditional i. Thus, this Appendix assumes some comfort with basic notions of the arithmetic of complex numbers.

We begin with the formal definition of the *natural logarithm* (also called the *Naperian* or the *hyperbolic* logarithm), $\ln x$:

$$\ln x \equiv \int_1^x \frac{dt}{t}, \tag{4A.2}$$

where the t in the integrand of eq. (4A.2) is a *dummy variable* of integration, and where three special values of the natural logarithm are noted:

$$\ln 1 = 0, \qquad \ln 0 = -\infty, \qquad \ln e = 1. \tag{4A.3}$$

The number e is defined as:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818284 \ldots. \tag{4A.4}$$

In view of the properties (4A.3), the Taylor series representation (see eq. (4.1)) of the natural logarithm is defined in terms of an argument that is centered around the value $a = 1$ (for x $\neq -1$ and $|x| \leq 1$):

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots. \tag{4A.5}$$

Further, the natural logarithm is related to the *common* or *Briggs* logarithm, which we colloquially call the *logarithm to base 10*, by

$$\ln x = (\ln 10)(\log_{10} x) \cong 2.303 \log_{10} x. \tag{4A.6}$$

The *exponential function* is defined as the inverse of the natural logarithm, that is, $x = \ln y$ if $y = e^x$. The Taylor series for the exponential function is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots . \tag{4A.7}$$

The results that now follow are obtained by formal manipulation from this Taylor series. For example, from eq. (4A.7) it can be shown that

$$e^{x+y} = 1 + (x+y) + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \cdots + \frac{(x+y)^n}{n!} + \cdots = e^x e^y, \tag{4A.8}$$

and that for complex numbers a

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \cdots + \frac{(ax)^n}{n!} + \cdots = (e^x)^a. \tag{4A.9}$$

Further, building on the result (4A.9), we can confirm the formula (4A.1) that Feynman found remarkable, which is known as the *De Moivre Theorem*:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = \cos x + i \sin x. \tag{4A.10}$$

In the last step of eq. (4A.10) we are recognizing the standard Taylor series expansions of the trigonometric functions that appear as the middle terms in that equation. Further,

$$e^{-ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = \cos x - i \sin x, \tag{4A.11}$$

so that from eqs. (4A.10) and (4A.11) we find we can write the trigonometric functions as:

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{4A.12}$$

We are now in a position to write down relations for the hyperbolic functions by replacing x by ix, and recalling the definition of i, so that:

$$\cosh x = \frac{1}{2}\left(e^{-x} + e^{x}\right) = \cos(ix), \quad \sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) = -i \sin(ix). \tag{4A.13}$$

It also follows from eqs. (4A.13) that

$$\cosh(ix) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos x, \ \ \sinh(ix) = \frac{1}{2}\left(e^{ix} - e^{-ix}\right) = i \sin x. \tag{4A.14}$$

While the structure and appearance of the trigonometric and hyperbolic functions appear to be very similar, their behavior is not. The trigonometric functions are periodic, with period 2π, and their values are always bounded by ± 1, that is, $-1 \leq (\sin x, \cos x) \leq 1$. The hyperbolic cosine increases monotonically for both positive and negative values of its argument, while the hyperbolic sinusoid is asymmetric about the origin and so approaches $-\infty$ as $x \to -\infty$. Oh, what a difference an i makes! We show further details of all of the elementary transcendental functions in Table 4A.1.

Table 4A.1 Behavioral features of the elementary transcendental functions.

$f(x)$	Value at $x = 0$	Behavior as $x \to \infty$	Behavior of $f(x)$
$\sin x$	0	$\lvert \sin x \rvert \leq 1$	Oscillates continuously between ± 1
$\cos x$	1	$\lvert \cos x \rvert \leq 1$	Oscillates continuously between ± 1
e^x	1	$\to \infty$	Uniformly increases as $(x > 0) \to \infty$
$\sinh x$	0	$\to \infty$	Uniformly increases as $(x > 0) \to \infty$; Uniformly decreases as $(x < 0) \to -\infty$
$\cosh x$	1	$\to \infty$	Uniformly increases as $x \to \pm\infty$
$\ln x$	$-\infty$	$\to \infty$	Uniformly increases as $(x > 0) \to \infty$
$\log_{10} x$	$-\infty$	$\to \infty$	Uniformly increases as $(x > 0) \to \infty$

Finally, some derivatives and integrals of the elementary transcendental functions are:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$
$$\frac{d^2}{dx^2}\sin x = -\sin x, \qquad \frac{d^2}{dx^2}\cos x = -\cos x. \tag{4A.15}$$

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d^n}{dx^n}e^x = e^x. \tag{4A.16}$$

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \frac{d}{dx}\cosh x = \sinh x,$$
$$\frac{d^2}{dx^2}\sinh x = \sinh x, \qquad \frac{d^2}{dx^2}\cosh x = \cosh x. \tag{4A.17}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}, \qquad \int \ln x\, dx = x\ln x - x. \tag{4A.18}$$

4.10 References

M. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematical Series 55, National Bureau of Standards, Washington, D.C., 1964.

A. H-S. Ang and W. H. Tang, *Probability Concepts in Engineering Planning and Design*, Vol. 1, John Wiley, New York, 1975.

R. A. Becker, *Introduction to Theoretical Mechanics*, McGraw-Hill, New York, 1954.

J. C. Burkill, *A First Course in Mathematical Analysis*, Cambridge University Press, London and New York, 1967.

K. C. Crandall and R. W. Seabloom, *Engineering Fundamentals in Measurements, Probability, Statistics, and Dimensions*, McGraw-Hill, New York, 1970.

C. L. Dym and E. S. Ivey, *Principles of Mathematical Modeling*, 1st Edition, Academic Press, New York, 1980.

C. E. Ebeling, *An Introduction to Reliability and Maintainability Engineering*, McGraw-Hill, New York, 1997.

R. M. Felder and R. W. Rousseau, *Elementary Principles of Chemical Processes*, John Wiley, New York, 2000.

R. Haberman, *Mathematical Models*, Prentice-Hall, Englewood Cliffs, NJ, 1977.

J. P. Holman, *Experimental Methods for Engineers*, McGraw-Hill, New York, 1971.

R. W. Hornbeck, *Numerical Methods*, Quantum, New York, 1975.

D. Huff, *How to Lie with Statistics*, Norton, New York, 1954.

L. B. W. Jolley, *Summation of Series*, Dover Publications, New York, 1961.

M. A. Lavrent'ev and S. M. Nikol'skiĭ, "Analysis," in A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev (Eds.), *Mathematics: Its Content, Methods, and Meaning*, Vol. I, MIT Press, Cambridge, MA, 1963.

C. Lipson and N. J. Sheth, *Statistical Design and Analysis of Engineering Experiments*, McGraw-Hill, New York, 1973.

W. Mendenhall and T. L. Sincich, *Statistics for Engineering and the Sciences*, 4th Edition, Prentice-Hall, Englewood Cliffs, NJ, 1995.

B. O. Pierce and R. M. Foster, *A Short Table of Integrals*, 4th Edition, Ginn and Company, Boston, MA, 1956.

H. J. Schenck, Jr., *Theories of Engineering Experimentation*, McGraw-Hill, New York, 1968.

F. W. Sears and M. W. Zemansky, *Modern University Physics*, Addison-Wesley, Reading, MA, 1960.

H. D. Young, *Statistical Treatment of Experimental Data*, McGraw-Hill, New York, 1962.
H. Ziegler, *Mechanics*, Vol. 1, Addison-Wesley, Reading, MA, 1965.

4.11 Problems

4.35. Estimate the error made in approximating $y(x) = \sin x$ with a Taylor's formula to $n = 4$ by evaluating the remainder R_5.

4.36. Do the statements that $\sin x \ll 1$ and $\tan x \ll 1$ produce similar approximations? Confirm and explain your answer.

4.37. The readings of an old-fashioned analog voltmeter—it has dials, not digital readouts!—are subject to some systematic error where all of its readings are too large. The magnitude of the error has been found to vary linearly from 1 V at a dial reading of 5 V to 4 V at a dial reading of 80 V.

(a) What are the correct voltages for dial readings of 80, 100, 50, 1, 35, and 10 V?
(b) What is the percentage error for each of the six (6) readings in part (a)?

4.38. (a) Is it possible to have a set of measurements that are precise but not accurate? Explain.
(b) Is it possible to have a set of measurements that are accurate but not precise? Explain.

4.39. (a) Write the Taylor series expansion for e^x about $x = 0$.
(b) Calculate $e^{0.5}$ to five significant figures using the first four terms of the series found it part (a).

4.40. (a) What percentage error was incurred in the calculation of part (b) of Problem 4.39 if the "true value" of $e^{0.5}$ is 1.6487?
(b) Use the Taylor remainder (eq. (4.5)) to calculate the error in $e^{0.5}$ after only four terms. Is the error calculated in part (a) of this problem acceptable? Explain.

4.41. Evaluate the following function by hand (no calculators or computers, please) for $x = 4$:

$$\left(1 + \frac{2}{x}\right)^{1/4}.$$

4.42. How does an observer know when enough is enough, that enough measurements have been taken?

4.43. Make a list of five new (i.e., not found in the text) examples of systematic errors.

4.44. Make a list of five new (i.e., not found in the text) examples of random errors.

4.45. The resistance of a resistor, R, is made by passing several currents, I, through it and measuring the corresponding voltage drops, V, and currents with imprecise, analog meters. The resulting data are:

$x_i = V(V)$	10	20	30	40	50	60	70	80
$y_i = I(A)$	0.8	1.1	2.5	4.2	4.3	4.7	5.8	6.4

(a) What kinds error will be found in the data?
(b) Assuming that $V = IR$, plot the data (by hand!) and "eyeball" in the best-fit line for that data.

4.46. Use the method of least squares to plot a V versus I curve for the data of Problem 4.45. How does it compare with the "eyeball" result of Problem 4.45?

4.47. The data presented below comprise 100 readings of noise levels taken 6 mi away from an airport, taken late in an evening at 15 s intervals. Find the mean, median, and standard deviation of these data.

Observed Decibel Values (dB), $n = 100$

50	50	53	48	45	51	57*	75*	85*	82*
75*	71*	65*	61*	60*	60*	55*	55*	51	50
49	49	48	51	49	54	48	48	47	49
49	49	49	49	48	47	50	49	48	49
47	48	48	50	50	54	48	47	47	48
48	49	48	47	50	49	48	48	48	48
48	48	52	50	53	49	49	48	49	47
49	55	51	50	49	48	49	45	48	50
50	51	49	50	47	47	47	47	47	47
48	50	49	49	49	49	49	49	56	49

4.48. The starred numbers in the data of Problem 4.47 are readings taken while an aircraft was flying directly overhead. If these data are deleted, what are the mean, median, and standard deviation of the remaining 88 data points?

4.49. Draw (a) a histogram of all of the data of Problem 4.47 and (b) a continuous curve of the number of readings as a function of the measured noise level.

4.50. Determine a *far-field approximation* of the function $f(r)$ given below as a binomial expansion for values of $r \gg a$.

$$f(r) = \sqrt{a^2 + r^2}.$$

4.51. The electric potential, V_e, at a distance, r, along the axis of revolution of a disk of radius a is given by

$$V_e = \frac{q}{2\pi a^2 \varepsilon_o}(\sqrt{a^2 + r^2} - r),$$

where q is the total charge that is distributed uniformly over the surface of the disk and ε_0 is the permittivity constant. Using the results of Problem 4.50, find a far-field approximation for the electric potential for values of $r \gg a$.

4.52. Compare the minimum number of terms kept in the binomial expansions of the solutions to Problems 4.50 and 4.51. Are those numbers the same, or not? Why are those numbers the same, or not?

4.53. Suppose we need to calculate the radial extension or deflection w of a very thin, spherical balloon, meaning that the sphere's radius extends from R to $R + w$ as the balloon is pressurized. It is made of an elastic material. A colleague finds a textbook that shows a formula for the pressure, p, that looks reasonable:

$$\frac{w}{R} = \frac{pR}{Eh},$$

where h is the balloon's wall thickness, and E is the modulus of the material of which the sphere is made. Is this equation dimensionally consistent?

4.54. Analyze the limit behavior of the equation presented in Problem 4.53 as the pressure, modulus, radius, and thickness both go to zero and become infinitely large. Does this limit behavior conform with your intuitive estimate of what should happen?

4.55. Use the equation in Problem 4.53 to derive an estimate of the magnitude of the pressure, p, as a fraction of the modulus, E. Estimate the pressure fraction for a thin-walled sphere, for which $h/R \ll 1$.