7
Modeling Free Vibration

We now turn to modeling vibration, the behavior of something moving back and forth, to and fro, usually in an evident rhythmic pattern. Vibration not only occurs all around us, but within us as well, as noted in 1965 by a well-known British mechanical engineer, R. E. D. Bishop:

After all, our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and our larynges vibrate. The light waves which permit us to see entail vibration. We move by oscillating our legs. We cannot even say 'vibration' properly without the tip of the tongue oscillating. And the matter does not end there—far from it. Even the atoms of which we are constituted vibrate.

Other vibratory phenomena that come to mind are pendulums, clocks, conveyor belts, machines and engines, buildings subjected to a broad array of moving forces (e.g., pedestrians, air conditioners, elevators, wind, earthquakes), as well as tides and seasons. Clearly, we could go on. But the more interesting questions for us are: Do these diverse instances of vibration have anything in common? If so, what? How do we model their common features? **Why?**

We devote most of this chapter to modeling a well-known "golden oldie," the swinging or vibrating pendulum. It provides a familiar platform upon which we can lay out a number of modeling strategies. Then we will provide a few examples of freely vibrating phenomena. We will also illustrate how the mathematics of free vibration can be used to model *stability* phenomena. In Chapter 8 we will provide some more examples and then go on to model forced vibration.

7.1 The Freely-Vibrating Pendulum–I: Formulating a Model

Given? We will now model the free vibration of a pendulum, starting with some experimental results and using dimensional analysis, some basic physics, and some basic mathematics (e.g., linearity, second-order differential equations) to model that motion.

7.1.1 Some Experimental Results

How? We started by building some very simple pendulums in the laboratory, each consisting of a lead-filled wooden ball suspended from a stand by an ordinary piece of string. A basic schematic of the laboratory set-up is shown in Figure 7.1. The balls were initially held at rest at some angle, θ_0, and then they were let go to swing back and forth until they all stopped moving. As each pendulum swung, we measured its *period of free vibration*, the time T_0 it takes to swing through two complete arcs (from $\theta = \theta_0$ to $\theta = -\theta_0$ and back again). The periods of vibration were measured with photoelectric cells that were placed at the lowest point on the pendulum arc ($\theta = 0$) and were in turn connected to digital counters operating with a gated pulse. The counters were turned on by the first passing of the pendulum and then off again at the second passing, thus providing a direct read of one-half of the period T_0.

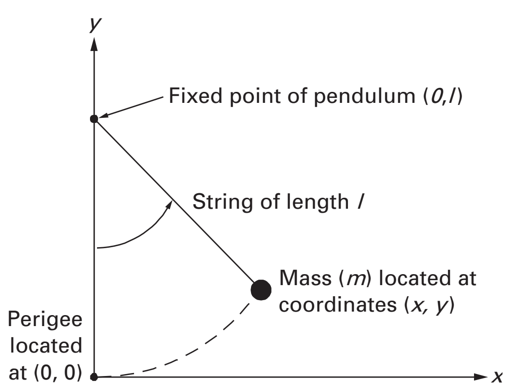

Figure 7.1 The geometry of a planar pendulum. Note that the origin of the coordinate system is located at the pendulum's *perigee*, the lowest point of its arc.

Table 7.1 The dependence of the period, T_0, of a freely-vibrating pendulum on its initial amplitude of vibration, θ_0. The mass is 390 gm and the string length is 276 cm.

θ_0(deg)	θ_0(rad)	T_0 **measured (sec)**	(T_0 **measured)/(3.372)**
8.34	0.1456	3.368	1.00
13.18	0.2300	3.368	1.00
18.17	0.3171	3.372	1.00
23.31	0.4068	3.372	1.00
28.71	0.5011	3.390	1.01
33.92	0.5920	3.400	1.01
39.99	0.6980	3.434	1.02
46.62	0.8137	3.462	1.03

The experiments were done with two different masses (237 gm and 390 gm), each of which was hung from strings of two different lengths (276 cm and 226 cm). The experimental data thus obtained are shown in Tables 7.1 and 7.2; note that each data point shown represents the average of five measured values. Thus, the data presented result from a consistent, repeatable experiment. The data in Table 7.1, for the larger mass (390 gm) and the shorter string (276 cm), show how the period, T_0, varies with different starting values of θ_0. We see that the period varies with the initial starting angle, θ_0, but the dependence is very weak and exceeds 1% only when $\theta_0 \geq 40°$.

Table 7.2 The dependence of the period, T_0, of a freely-vibrating pendulum on its length and on its mass. The data show a marked change with length, but virtually no change with mass.

	$m = 237$ gm	$m = 390$ gm
$l = 226$ cm	3.044 sec	3.058 sec
$l = 276$ cm	3.350 sec	3.372 sec

The data in Table 7.2 summarize the periods across the four possible combinations of mass and length that were available for the pendulums used in this experiment. This data suggest that the period varies very little, if at all, with mass: increasing the mass by some 65% from 237 gm to 390 gm changes the period by a fraction of 1%. On the other hand, increasing the length by 22% from 226 cm to 276 cm increases the period by approximately 10%. Thus, the data suggest that the free motion of a vibrating pendulum is *periodic*, and that the period of vibration does not depend on the pendulum's mass, but that it does depend on the pendulum's length.

Problem 7.1. Assume a hypothetical relationship, $T_0 = am^b$, for the dependence of the period of a pendulum on its mass. Determine the unknown parameters, a and b, using the data in Table 7.2. (*Hint*: Logarithms may be useful here.)

Problem 7.2. Assume a hypothetical relationship, $T_0 = cl^d$, for the dependence of the period of a pendulum on its length. Determine the unknown parameters c and d using the data in Table 7.2. (*Hint*: Logarithms may be useful here.)

7.1.2 Dimensional Analysis

We will now apply some dimensional analysis results to formalize the results we obtained in the laboratory. In Section 2.4.2 we used the Buckingham Pi theorem to determine that the period of vibration, T_0, of a pendulum was related to its length, l, and the gravitational acceleration, g [see the first of eq. (2.30)]:

$$T_0 = \Pi_1 \sqrt{\frac{l}{g}}. \tag{7.1}$$

Valid? Note that the pendulum's period does not depend on mass, a result supported by the data in Table 7.2, and that the constant, Π_1 is dimensionless. We can determine the value of Π_1 from the data given in Table 7.2. For the pendulum of length $l = 276$ cm, one measured value of the period is $T_0 = 3.372$ sec, so that with $g = 980$ cm/sec/sec,

$$\Pi_1 = \frac{3.372}{\sqrt{276/980}} \cong 6.35. \tag{7.2}$$

Is the number "6.35" in eq. (7.2) some new universal constant? Actually, no. Rather, it is an approximation of another well-known constant: $2\pi \cong 6.28$. Thus, substituting this judgment call about the constant into eq. (7.2) yields the final result,

$$T_0 = 2\pi \sqrt{\frac{l}{g}}. \tag{7.3}$$

Table 7.3 Calculated values of the period, T_0, of a freely-vibrating pendulum that provide support for the experimental data presented in Table 7.2.

l (cm)	T_0 (sec)
226	3.02
276	3.33

We can use eq. (7.3) to predict values of the period to match the remaining values displayed in Table 7.2, as shown in Table 7.3. The calculated predictions and the experimental data agree to within less than 1.5%. Thus, it seems that we have a pretty good model—determined from dimensional analysis and use of some experimental data—that works quite well and predicts the remaining experimental data, including both the period's dependence *on* length and its independence *of* mass. We will confirm the model (7.3) again before we're done with the pendulum. **Predict? Verified?**

7.1.3 Equations of Motion

We formulate the problem by writing the mathematical expression of a balance or conservation principle (see Section 1.3.3) from physics. The principle is Newton's second law: *The time rate of change of momentum is equal to the net force producing it; that momentum change is in the same direction as the net force.* Newton's second law is both a balance principle and a conservation principle: it reflects a balance of the forces acting on a particle or system, and it also reflects the conservation of momentum. Written as a balance principle (see Problems 7.3 and 7.4), Newton's second law in a plane is: **How?**

$$\sum F_x = m\frac{d^2x}{dt^2}, \tag{7.4a}$$

and

$$\sum F_y = m\frac{d^2y}{dt^2}, \tag{7.4b}$$

where $x(t)$ and $y(t)$ are the time-dependent coordinates of a mass, m, acted on by net forces $\sum F_x$ and $\sum F_y$, respectively.

We want to apply Newton's second law, commonly referred to as *equations of equilibrium*, to the pendulum depicted in Figure 7.1. The pendulum is simply a mass, m, attached to the end of a string of length, l. It swings in a plane from an attachment point with coordinates $(0, l)$ so that the origin of the coordinates coincides with the *perigee* or low point of the pendulum's arc. The coordinates (x, y) of the pendulum mass can be written in terms of the string length and the angle θ between the string and the y-axis:

$$x(t) = l \sin\theta(t), \tag{7.5a}$$

and

$$y(t) = l(1 - \cos\theta(t)), \tag{7.5b}$$

In Figure 7.2 we show a *free-body diagram* (FBD) of the two forces that act on the mass: the tension in the string, T, and the weight, mg, which acts due to the pull of gravity. Then we can identify the net forces along the coordinates from the FBD, so that eqs. (7.4) can then be written as *equations of motion*:

$$m\frac{d^2x}{dt^2} = \sum F_x = -T\sin\theta, \tag{7.6a}$$

and

$$m\frac{d^2y}{dt^2} = \sum F_y = T\cos\theta - mg. \tag{7.6b}$$

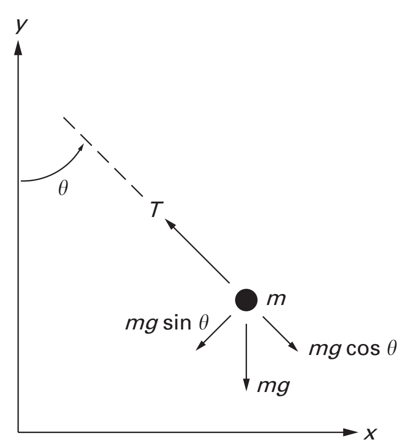

Figure 7.2 A *free-body diagram* (FBD) of the oscillating planar pendulum. It shows the two forces acting on the pendulum's mass, m, the string tension, T, and the weight, mg, and their components in the radial and tangential directions.

In principle, all we need to do now is integrate eqs. (7.6a–b) to find how the pendulum's coordinates vary with time, from which we can then find out whatever else we might want to know about the pendulum. However, life's not that easy, for a number of reasons. First, we don't know the tension in the string, T, so that the right-hand sides of both of eqs. (7.6a–b) are unknown. Second, since we have two equations with *three* unknowns—$x(t)$, $y(t)$, T—we are prompted to wonder how Newton's second law would look if rewritten in *radial* (along the string) and *tangential* (to the pendulum's arc) coordinates. In fact, those equations are **Improve?**

$$\sum F_{\text{radial}} = ml\left(\frac{d\theta}{dt}\right)^2, \tag{7.7a}$$

and

$$\sum F_{\text{tangential}} = ml\frac{d^2\theta}{dt^2}. \tag{7.7b}$$

Equation (7.7a) clearly displays the familiar centripetal acceleration. If we sum the forces in the FBD of Figure 7.2 in the radial and tangential directions, we would find that

$$T = ml\left(\frac{d\theta}{dt}\right)^2 + mg\cos\theta, \tag{7.8a}$$

and

$$ml\frac{d^2\theta}{dt^2} + mg\sin\theta = 0. \tag{7.8b}$$

Equations (7.8a–b) show two equations for two dependent variables, the tension, T, and the angle, θ. Equation (7.8b) is a single equation with a single unknown, θ, so it can in principle be solved on its own, which thus determines the location of the mass [see also eqs. (7.5a–b)]. Then the tension, T, can be obtained directly by substituting the newly-found θ into eq. (7.8a). We also note that eqs. (7.8a–b) are equivalent to eqs. (7.6a–b): both are representations of Newton's second law, eqs. (7.8a–b) written in radial and tangential coordinates (l, θ), eqs. (7.6a–b) in rectangular coordinates (x, y).

We further note that eqs. (7.8a–b) are decidedly nonlinear because the dependent variable $\theta(t)$ or its derivatives have an exponent different than 1. This is most obvious in eq. (7.8a) because of the centripetal acceleration (see Problem 7.5), but it is equally true of eq. (7.8b) because

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots . \tag{7.9}$$

As we noted in Section 1.3.4, the presence of such nonlinear terms means that superposition, one of the most powerful weapons in the arsenal of mathematics, is no longer available. We will return to this point in greater detail in Section 7.3.

Problem 7.3. Why do eqs. (7.4a–b) represent Newton's second law as a balance principle?

Problem 7.4. How would eqs. (7.4a–b) be written as a conservation principle?

Problem 7.5. Identify and explain *all* of the nonlinearities in eq. (7.8a).

7.1.4 More Dimensional Analysis

Valid? Are the dimensions of eqs. (7.8a–b) correct and consistent? Can we use dimensional information to further our understanding? In Table 7.4, we show (again, see Table 2.2) the pendulum variables expressed in terms of the fundamental dimensions of mass, length, and time. With this data, we can confirm (see Problem 7.6) that each of the terms in eqs. (7.8a–b) has the physical dimensions of *force*, or in terms of fundamentals, $(\mathsf{M} \times \mathsf{L})/\mathsf{T}^2$, which is appropriate for an equation of equilibrium. Further, we have satisfied the test that every stand-alone term in an equation has the same dimensions.

We now introduce a *scaling factor*, ω_0, that has, by definition, the dimensions of $1/\mathsf{T}$. The scaling factor also allows us to introduce a *dimensionless* time variable, τ, defined as

$$\tau = \omega_0 t. \tag{7.10}$$

Table 7.4 The fundamental dimensions of the six derived quantities chosen to model the oscillating pendulum.

Derived Quantities	Dimensions
Length (l)	L
Gravitational acceleration (g)	L/T^2
Mass (m)	M
Period (T_0)	T
Angle (θ)	1
String tension (T)	$(\mathsf{M} \times \mathsf{L})/\mathsf{T}^2$

Then the tangential equation of motion (7.8b) can be written as (see Problem 7.7)

$$l\omega_0^2 \frac{d^2\theta(\tau)}{d\tau^2} + g\sin\theta(\tau) = 0. \tag{7.11}$$

Hence, if we choose the scaling factor, ω_0, to be

$$\omega_0 = \sqrt{g/l}, \tag{7.12}$$

we can write the tangential equation of motion (7.11) in a rather elegant form that is *completely dimensionless*:

$$\frac{d^2\theta(\tau)}{d\tau^2} + \sin\theta(\tau) = 0. \tag{7.13}$$

Note that the dimensions of the scaling factor are reciprocal to the dimensions of the period of free vibration, T_0, and that eqs. (7.3) and (7.12) can be combined to eliminate the common radicand, thus yielding:

$$T_0 = \frac{2\pi}{\omega_0}. \tag{7.14}$$

Equation (7.14) strongly suggests that we should recognize that the scaling factor, ω_0, is actually the circular frequency of the pendulum, that is, the measure of the pendulum's periodicity expressed in radians per unit of time. **Use?**

Now that we have confirmed dimensional consistency and cast at least one of our equilibrium equations in an elegant, dimensionless form, can we learn anything else? We can. We start by observing that $|\sin\theta| \leq 1$. This means that the acceleration term in eq. (7.11) must also exhibit similar behavior: $|d^2\theta/d\tau^2| \leq 1$, which provides a time scale for the problem. To demonstrate this, consider the function:

$$\theta(\tau) = \theta_0 \cos\tau, \tag{7.15}$$

for which it follows that

$$\frac{d\theta(\tau)}{d\tau} = -\theta_0 \sin\tau \quad \text{and} \quad \frac{d^2\theta(\tau)}{d\tau^2} = -\theta_0\cos\tau. \tag{7.16}$$

which means that $\theta(\tau)$ and all of its derivatives with respect to τ have the same maximum amplitude θ_0.

If we choose to make our independent variable, t, dimensionless as we just did, are there any restrictions we need to place on its dimensionless counterpart, τ? No. Equations (7.10) and (7.12) tell us that

$$\tau = t\omega_0 = \frac{t}{1/\omega_0}, \tag{7.17}$$

which can be seen as a "verbal" or "conceptual equation":

$$\tau = \frac{\text{actual physical time}}{\text{a constant with dimensions of time}}. \tag{7.18}$$

Use? Equation (7.18) tells us that we get to choose how we make our equations dimensionless by choosing "a constant with dimensions of time" to match the problem of interest. If we are modeling something that takes years, the "constant" should be expressed in years. Then, small values of the dimensionless time, τ, would mean times of weeks, days, or even hours. Large values of the dimensionless time, τ, would mean times of decades, centuries, or even millennia.

Sometimes the "constant" is determined or dictated by the physics of the problem being investigated. For example, for a pendulum that is 1 m long, $\omega_0 = \sqrt{g/l} \cong 3.13\ \text{sec}^{-1}$, we would say that the system has a characteristic time of about one-third of a second—implying that the pendulum is moving rather fast. For a rather long pendulum, say $l = 98$ m, $\omega_0 = \sqrt{g/l} \cong 0.31\ \text{sec}^{-1}$ the system has a characteristic time of about 3 sec.

Problem 7.6. Identify the fundamental dimensions of each free-standing term in eqs. (7.8a–b) and confirm that each has net dimensions of force.

Problem 7.7. Substitute the dimensionless variable of eq. (7.10) into eq. (7.8b) to verify eq. (7.11).

7.1.5 Conserving Energy as the Pendulum Moves

Why? We now turn to a qualitative analysis of the behavior of solutions to the differential equations of motion (7.6) or (7.8). But we start not with the differential equations themselves, but with considerations of energy rooted in the basic physics. When the pendulum is swinging through its arc, it possesses kinetic energy and potential energy. As we will see, each of these energies may vary with position, but both are present and their sum will be a constant.

The kinetic energy, *KE*, is found from a familiar calculation:

$$KE = \frac{1}{2}m(\text{speed})^2. \tag{7.19}$$

The speed can be calculated in the usual way by differentiating the coordinates of mass [eqs. (7.5a–b)] with respect to time, to find that (see Problem 7.8)

$$KE = \frac{1}{2}m\left(l\frac{d\theta(t)}{dt}\right)^2 = \frac{1}{2}mgl\left(\frac{d\theta(\tau)}{d\tau}\right)^2. \tag{7.20}$$

The potential energy of the swinging mass, PE, is measured with respect to a datum through the origin of the coordinates ($x = 0, y = 0$) in another familiar calculation:

$$PE = mgy(t) = mgl(1 - \cos\theta(\tau)). \tag{7.21}$$

Then the total energy, $E(\tau)$, is found by adding eqs. (7.20) and (7.21):

$$E(\tau) = KE + PE = mgl\left[\frac{1}{2}\left(\frac{d\theta(\tau)}{d\tau}\right)^2 + (1 - \cos\theta(\tau))\right]. \tag{7.22}$$

How does the total energy vary with time? A straightforward differentiation shows that

$$\frac{dE(\tau)}{d\tau} = mgl\left[\frac{d^2\theta(\tau)}{d\tau^2} + \sin\theta(\tau)\right]\left(\frac{d\theta(\tau)}{d\tau}\right). \tag{7.23}$$

Equation (7.23) is a remarkable result! The term in the brackets is identical to the tangential equation of motion (7.8b). Thus, two lessons emerge. First, we recover the equation of motion of a system by differentiating its total energy. Second, if $\theta(t)$ is such that the equation of motion is satisfied, then *the total energy is conserved*:

$$\frac{dE(\tau)}{d\tau} = 0 \quad \text{and} \quad E(\tau) = E_0 = \text{constant}. \tag{7.24}$$

Can we determine this constant value of energy, E_0? We can by recognizing that we imparted some energy to the pendulum when we let it start swinging from a rest position θ_0. Thus, the initial potential energy is, in fact, the initial total energy:

$$PE(0) = mgy(0) = mgl(1 - \cos\theta_0) = E_0. \tag{7.25}$$

Problem 7.8. What is the speed of the pendulum mass expressed in polar coordinates? How does that relate to eq. (7.20)?

Problem 7.9. Can eq. (7.22) be simplified for small angles of oscillation? If so, how?

Problem 7.10. How would eq. (7.23) appear after the simplifications of Problem 7.9?

7.1.6 Dissipating Energy as the Pendulum Moves

Why? **How?**

Our discussion of the pendulum has thus far assumed it to be *ideal* in that no energy was lost. We now extend our model to include the effects of the *damping forces* that arise when motion is resisted by friction or air resistance. Damping or friction forces are generally assumed to be the result of *viscous damping* that is proportional to the speed of the object being analyzed (and slowed by the damping), with a constant of proportionality, c, called the *damping coefficient.* For a viscous damping force we have

$$F_{\text{damping}} = -c(\text{velocity}), \tag{7.26}$$

where c is a positive constant with dimensions of force per unit velocity or M/T. The minus sign in eq. (7.26) reflects the fact that the viscous damping slows or retards the pendulum motion by opposing it. For the swinging pendulum, the retarding force would act tangentially, so that the friction force would appear in a suitably modified version of the tangential equation of motion (7.8b):

$$ml\frac{d^2\theta}{dt^2} + cl\frac{d\theta}{dt} + mg\sin\theta = 0. \tag{7.27}$$

How does the inclusion of the damping force affect the energy of the pendulum? The forms of the kinetic and potential energies are unchanged by the damping force, so that the total energy can be written as before [eq. (7.22)], except in terms of real time, t:

$$E(t) = \frac{1}{2}ml^2\left(\frac{d\theta(t)}{dt}\right)^2 + mgl(1-\cos\theta(t)). \tag{7.28}$$

The time rate of change of the energy is [again, much as before in eq. (7.23)]

$$\frac{dE(t)}{dt} = \left[ml^2\frac{d^2\theta(t)}{dt^2} + mgl\sin\theta(t)\right]\left(\frac{d\theta(t)}{dt}\right),$$

which in view of eq. (7.27) can be cast as

$$\frac{dE(t)}{dt} = -cl^2 \left(\frac{d\theta(t)}{dt}\right)^2. \tag{7.29}$$

Equation (7.29) shows that the pendulum's total energy steadily decreases with time.

We can take this a step further with the following argument. The energy of an ideal pendulum as it begins from rest is entirely potential energy, and its energy is entirely kinetic when the pendulum swings through its perigee (because the origin of our coordinate system is located at the perigee). Thus, on average, the kinetic and potential energies are approximately the same, even in the presence of all but the most severe damping. To the extent this argument is reasonable, we can approximate the total energy of the pendulum—whether damped or not—as twice the kinetic energy:

$$E(t) \cong ml^2 \left(\frac{d\theta(t)}{dt}\right)^2 \tag{7.30}$$

Now we can eliminate the term $(d\theta/dt)^2$ between eqs. (7.29) and (7.30) to obtain a differential equation for the energy $E(t)$:

$$\frac{dE(t)}{dt} = -(c/m)E(t). \tag{7.31}$$

Note that the dimensions of (c/m) are force per unit velocity per unit mass or 1/T. Thus, eq. (7.31) is dimensionally consistent. **Verify?**

Equation (7.31) is also a first-order differential equation with constant coefficients, like the models developed in Chapter 5. Thus, the solution to eq. (7.31) is

$$E(t) = E_0 e^{-(c/m)t}. \tag{7.32}$$

Equation (7.32) shows that the total energy decays exponentially from its initial maximum value, E_0, imparted by the pendulum's initial position. The rate at which the energy decays depends on a *characteristic decay time*, m/c. The characteristic decay time has the proper dimensions, and its precise value (measured in seconds, days, or centuries) will depend on the particular pendulum being modeled. However, we can calculate the energy decay as a function of time measured as a multiple of the characteristic decay time. Table 7.5 shows us that the energy of a damped pendulum is halved in a time equal to $0.69(m/c)$—which is a useful indicator of energy decay time. **Use?**

We note in closing this part of the discussion that we have already learned a lot about the swinging pendulum—and we have determined that information without knowing the specific form of $\theta(t)$ and without

Table 7.5 The decay of the total energy of an oscillating pendulum expressed in multiples of the characteristic decay time, m/c.

Time	Energy
$t = 0$	$E(t) = E_0$
$t = 0.10(m/c)$	$E(t) = 0.905E_0$
$t = 0.69(m/c)$	$E(t) = 0.500E_0$
$t = 1.00(m/c)$	$E(t) = 0.368E_0$
$t = 5.00(m/c)$	$E(t) = 0.007E_0$

solving the differential equations of motion that describe the pendulum's arc. Note, too, that we have not had to distinguish between linear and nonlinear models of the pendulum's behavior, so that the results already obtained—and the methods used to obtain them—are valid for a relatively large class of problems. We will go on to solve the differential equations for the linear model of the pendulum in Section 7.2 and for its nonlinear model in Section 7.5.

7.2 The Freely-Vibrating Pendulum–II: The Linear Model

Why? In Section 7.3 we will come to know the linear model of the pendulum as the ubiquitous *spring-mass oscillator*. But now we want to know: How does a nonlinear model become linear? What do the solutions to linear models look like?

7.2.1 Linearizing the Nonlinear Model

How? We turn a nonlinear model into a linear model by the process of *linearization* in which magnitudes and behaviors are assumed to be sufficiently small in some sense that their products can be neglected. This may not always be possible, and it must be done carefully even when it is possible, because some phenomena are so inherently nonlinear that they can never be linearized. There are nonlinear terms in the pendulum's radial and tangential equations of motion (7.8), which we write here in terms of the dimensionless time, τ, defined in eq. (7.10) and with the nonlinear terms underlined:

$$T = mg\left[\underline{\left(\frac{d\theta(\tau)}{d\tau}\right)^2} + \underline{\cos\theta(\tau)}\right], \tag{7.33a}$$

and

$$\frac{d^2\theta(\tau)}{d\tau^2} + \underline{\sin\theta(\tau)} = 0. \tag{7.33b}$$

Now let us assume that the angle of the pendulum can be written as

$$\theta(\tau) = \theta_0 f(\tau), \tag{7.34}$$

where $f(\tau)$ is a function whose absolute value is such that $|f(\tau)| \leq 1$. Then

$$\theta_0 = \max|\theta(\tau)|. \tag{7.35}$$

We can identify θ_0 as the *amplitude* of the pendulum's motion that indicates the magnitude of the pendulum's swings. We want to define just how large that amplitude may be, whether it is *small* or *large*, which means that we must provide a reference against which we can meaningfully measure *small* and *large*. We do that by referring back to the Taylor series for the trigonometric functions given in Section 4.1.2, now written in terms of the amplitude θ_0:

$$\sin\theta_0 = \theta_0 - \frac{\theta_0^3}{3!} + \frac{\theta_0^5}{5!} - \frac{\theta_0^7}{7!} + \cdots \cong \theta_0 + O(\theta_0^3) \tag{7.36a}$$

$$\cos\theta_0 = 1 - \frac{\theta_0^2}{2!} + \frac{\theta_0^4}{4!} - \frac{\theta_0^6}{6!} + \cdots \cong 1 + O(\theta_0^2) \tag{7.36b}$$

In writing these results we have again (see the last two paragraphs of Section 4.1.2) assumed that the angle θ_0, expressed in radians, is a number that is small compared to 1. In eqs. (7.36) we have also introduced the *order* notation, $O(\theta_0^2)$, that indicates the lowest exponent on the remaining, unwritten terms in the series that represent the difference between a linear approximation, the first term in each series, and the function being approximated. The question of how many terms need to be retained in these series is answered simply: What level of precision is required of the model we are building? It is easy enough to show (see Problems 7.11–7.14) that we can approximate the sine and cosine functions by their *linear approximations* for angles $|\theta_0| \leq \pi/6 = 30°$ as follows: **Valid?**

$$\sin\theta_0 \cong \theta_0 \quad \text{percent error} \sim 5\% \tag{7.37a}$$

$$\cos\theta_0 \cong 1 \quad \text{percent error} \sim 15\% \tag{7.37b}$$

With the approximation (7.37a), we can immediately linearize the tangential equation of motion (7.33b):

$$\frac{d^2\theta(\tau)}{d\tau^2} + \theta(\tau) = 0. \tag{7.38}$$

A similar linearization of the cosine in the radial equation of motion (7.33a) produces the result that

$$T = mg\left[\underline{\left(\frac{d\theta(\tau)}{d\tau}\right)^2} + 1\right], \tag{7.39}$$

which still retains a nonlinear term. However, in the light of eq. (7.34) and the discussion of Section 7.1.4, it is easy enough to show (see Problem 7.15) that the values of $\theta(\tau)$ and its derivatives with respect to τ are all of the same order of magnitude or size. The underlined quadratic term in eq. (7.39) can then be neglected compared to 1, so the linearized model of the pendulum produces a *constant tension*:

$$T \cong mg. \tag{7.40}$$

We close this discussion of linearization by noting that notwithstanding the argument just made about the derivatives of $\theta(\tau)$ with respect to τ, we cannot assume that $\theta(t)$ and its derivatives with respect to the real time, t, are of the same order of magnitude. That assumption is valid *only* with respect to the dimensionless forms discussed.

Problem 7.11. How many terms of the series (7.36a) are needed to calculate $\sin\theta_0$ to a precision of 1% for angles $|\theta_0| \leq \pi/6 = 30°$? To 2%? To 5%?

Problem 7.12. How many terms of the series (7.36b) are needed to calculate $\cos\theta_0$ to a precision of 1% for angles $|\theta_0| \leq \pi/6 = 30°$? To 2%? To 5%?

Problem 7.13. Explain any differences between the answers to Problems 7.11 and 7.12.

Problem 7.14. How does a computer produce values of the "trig" and other transcendental functions?

Problem 7.15. Show (and explain) why the derivatives of eq. (7.34) with respect to τ are all of the same magnitude or size.

7.2.2 The Differential Equation $md^2x/dt^2 + kx = 0$

How do we determine the function $\theta(\tau)$ that satisfies and thus solves eq. (7.38)? First, to be more general, let us return that equation to its dimensional form, **How?**

$$ml\frac{d^2\theta(t)}{dt^2} + mg\theta(t) = 0. \tag{7.41}$$

To be still more general, we write eq. (7.41) in the equivalent form (see Problem 7.16) of

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0, \tag{7.42}$$

which is the classical equation for a simple *spring-mass oscillator*, which we will begin to discuss in some detail in Section 7.3 and with great generality in Chapter 8. In the meantime, we can safely refer to m as the (constant) *mass* of the oscillator, k as its (constant) *stiffness*, and $x(t)$ as its displacement (or movement or deflection). It is clear that if we can solve eq. (7.42) we obtain a solution to eq. (7.38).

Equation (7.42) is a *homogeneous, second-order, linear* differential equation that has *constant coefficients*, k and m. Guided by the discussion of Section 5.2.2, we assume a solution to eq. (7.42) in the form

$$x(t) = Ce^{\lambda t}, \tag{7.43}$$

which when substituted into eq. (7.42) leads to the *characteristic equation* that defines the constant, λ,

$$m\lambda^2 + k = 0. \tag{7.44}$$

Equation (7.44) has two solutions,

$$\lambda_{1,2} = \pm\sqrt{-1}\sqrt{\frac{k}{m}} \equiv \pm j\omega_0, \tag{7.45}$$

where we have now noted that $j = \sqrt{-1}$ and have redefined the scaling factor, ω_0, as

$$\omega_0 \equiv \sqrt{\frac{k}{m}} \tag{7.46}$$

Since eq. (7.42) is of second order, we expect that it will have two solutions, each corresponding to the two values of λ defined by eq. (7.45):

$$x(t) = C_1e^{j\omega_0 t} + C_2e^{-j\omega_0 t}. \tag{7.47}$$

These general forms of the homogeneous solutions are quite valid. However, guided by the "most remarkable formula" presented in Section 4.9, we can (see Problem 7.17) rewrite the solution (7.47) in terms of the standard trigonometric functions:

$$x(t) = B_1 \cos \omega_0 t + B_2 \sin \omega_0 t, \tag{7.48}$$

where B_1 and B_2 are two arbitrary constants that are entirely equivalent to the constants in eq. (7.47). It is also easily verified by direct substitution (Problem 7.18) that eq. (7.48) is a solution to eq. (7.42).

Equation (7.48) is called the *homogeneous solution* of eq. (7.42) because it solves a differential equation that has no forcing function on its right-hand side. Equation (7.48) is also called the *transient solution* because it actually represents the initial conditions that initiate the pendulum's motion. Thus, if $x(0) = x_0$ and $dx(0)/dt = \dot{x}_0$, it is easily shown (Problem 7.19) that

Use?

$$x(t) = x_0 \cos \omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t, \tag{7.49}$$

As we will further describe in the next section, the motion described by eq. (7.49) is periodic and would go on indefinitely for an ideal pendulum that experiences no damping. However, for a damped pendulum, this initial motion will be damped out, which is why it is called the "transient solution."

Problem 7.16. What is the effective spring stiffness, k, for the simple pendulum? Are its dimensions proper, for the pendulum itself and as a stiffness?

Problem 7.17. Use "the most remarkable formula" in mathematics to show how eq. (7.47) becomes eq. (7.48).

Problem 7.18. Substitute the solution (7.48) into eq. (7.42) and confirm that it is a correct solution.

Problem 7.19. Determine the constants, B_1 and B_2, in eq. (7.48) for the initial conditions $x(0) = x_0$ and $dx(0)/dt = \dot{x}_0$.

7.2.3 The Linear Model

Use? Returning now to the linear model of the pendulum, we can straightforwardly cast eq. (7.49) into the dimensionless notation of the pendulum (see Problem 7.20):

$$\theta(\tau) = \theta_0 \cos \tau + \dot{\theta}_0 \sin \tau, \tag{7.50}$$

where we can now identify θ_0 and $\dot{\theta}_0$ as, respectively, the initial location and the initial speed with which the pendulum is set in motion. These initial parameters are entirely independent, so that they can be specified separately. Thus, to drop a pendulum from a fixed angle, θ_0, but with no initial speed, the transient solution would be

$$\theta(\tau) = \theta_0 \cos \tau. \tag{7.51}$$

On the other hand, to launch the pendulum from the origin, $\theta_0 = 0$, with a specified initial speed, $\dot{\theta}_0$, the transient solution would take the form

$$\theta(\tau) = \dot{\theta}_0 \sin \tau. \tag{7.52}$$

Since we are solving a linear problem, superposition applies (see Section 1.3.4), and the general solution (7.50) is simply the sum of the two solutions (7.51) and (7.52).

In Section 4.9 we noted that the elementary trigonometric functions are periodic: the functions sin τ and cos τ have the same value when their arguments are increased by 2π, that is,

$$\cos(\tau + 2\pi) = \cos \tau \quad \text{and} \quad \sin(\tau + 2\pi) = \sin \tau. \tag{7.53}$$

In physical time t, then, the value of $\theta(t)$ repeats at time intervals such that

$$t = \frac{2\pi n}{\omega_0} = nT_0, \quad n = 1, 2, 3, \ldots . \tag{7.54}$$

Hence, T_0 is (again) the period of the pendulum motion and ω_0 its circular frequency, measured in radians per unit time. We can also define a frequency f_0 with units of $(\text{time})^{-1}$ or hertz (Hz), named after a famous acoustician, Heinrich Rudolf Hertz (1857–1894):

$$f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi} \tag{7.55}$$

One last observation about the results just described: the period of the vibrating pendulum, T_0, depends *only* on the physical properties of the pendulum and *not at all* on the amplitude of the oscillation. The uncoupling of the amplitude from the period, like the applicability of the principle of superposition, is another defining characteristic of linear models of vibration.

Problem 7.20. Show how the solution (7.49) becomes the solution (7.50) for initial conditions $\theta(0) = \theta_0$ and $d\theta(0)/dt = \dot{\theta}_0$.

7.3 The Spring-Mass Oscillator–I: Physical Interpretations

Why? **How?** We now explore some physical interpretations of the linear model just developed. The more general form, eq. (7.42), is an equation of equilibrium, which means that its physical dimensions are of force or $\mathsf{F} = \mathsf{ML}/\mathsf{T}^2$. Since $x(t)$ is the oscillator displacement and has the dimensions of length or L, the stiffness, k, must have the dimensions of force per unit length or F/L. Thus, the equation (7.42) represents a balance of an inertial force with a spring force. Further, our everyday experience with springs confirms Hooke's law, which states that a spring exerts a restoring force that is directly proportional to the amount that it is stretched or compressed, that is,

$$F_{\text{spring}} = kx(t). \tag{7.56}$$

Note that the sign of the spring force changes with the sign of the displacement, so that extending a spring ($x > 0$) produces a positive, tensile force that tends to return it to its original length, while compressing the spring ($x < 0$) produces a negative, compressive force that also tends to restore the spring to its original length.

How does this work for the pendulum? A slight rewriting of eq. (7.41) shows that

$$m\frac{d^2\theta(t)}{dt^2} + \left[k = \frac{mg}{l}\right]\theta(t) = 0. \tag{7.57}$$

Thus, we see that the pull of gravity acts just like a spring, exerting a larger restoring force as the pendulum angle increases.

Another reflection of this behavior can be seen if we examine the energy of the spring-mass oscillator. If we multiply eq. (7.42) by the oscillator speed, $dx(t)/dt$, we find

$$\left[m\frac{d^2x(t)}{dt^2} + kx(t)\right]\frac{dx(t)}{dt} = 0. \tag{7.58}$$

Now, both terms in eq. (7.58) are total derivatives. Therefore, we can integrate this equation to obtain

$$\frac{1}{2}m\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}k(x(t))^2 = E_0. \tag{7.59}$$

Thus, by inverting the process by which we identified the pendulum's total energy in Section 7.1.5, we have here derived the energy of the spring-mass oscillator and showed that it, too, is the sum of the kinetic and potential energies. Further, as we know from the pendulum and can easily demonstrate (see Problems 7.21–7.23) with the solution (7.49), the energy moves back and forth from being entirely kinetic energy when the pendulum is at its perigee to a position when it is entirely potential energy, that is, at its maximum amplitude. This means that each of the two elements in the spring-mass system acts as an *energy-storage element*: the spring stores (and releases) potential energy, while the mass stores (and gives up) kinetic energy. **Predict? Use?**

Problem 7.21. Calculate the *kinetic* energy of a spring-mass oscillator released from a rest position $x(0) = x_0$ initially and at time intervals, $T_0/4$, $T_0/2$, $3T_0/4$, and T_0.

Problem 7.22. Calculate the *potential* energy of a spring-mass oscillator released from a rest position $x(0) = x_0$ initially and at time intervals, $T_0/4$, $T_0/2$, $3T_0/4$, and T_0.

Problem 7.23. What fractions of the total energy are the kinetic and potential energies at time intervals, $T_0/4$, $T_0/2$, $3T_0/4$, and T_0? (*Hint*: Use the results of Problems 7.21 and 7.22!)

7.4 Stability of a Two-Mass Pendulum

In our brief review of the elementary transcendental functions (in Section 4.9), we saw that trigonometric and hyperbolic functions are closely related. The arithmetic difference between the two is traceable to the j factor in the argument of the exponential function. Their behaviors differ as well, with the trigonometric functions showing bounded periodicity and the hyperbolic functions showing exponential growth or decay. The change from periodic to exponential arithmetic behavior typically signals a change in physical behavior from a stable, bounded configuration to unstable, unbounded exponential growth. The transition from bounded trigonometric behavior to unbounded exponential behavior occurs when a model parameter passes through a critical value. We will illustrate this transitional behavior for a two-mass pendulum. **Why?**

Consider the vertically-arrayed dumbbell shown in Figure 7.3. If set absolutely still in a perfectly vertical alignment, it conceivably could remain in that precarious position. However, in the normal course of events, if the dumbbell is let go and starts to swing, we would expect that its final position—and its behavior in arriving at that position—will depend very

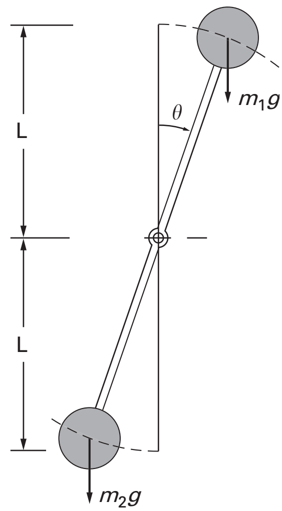

Figure 7.3 A schematic of a dumbbell, a two-mass pendulum. Its initial state has a mass, m_1 on top, and a mass, m_2 on the bottom. The stability of this state is dependent on the relative magnitudes of the two masses.

much on the relative sizes of the masses, m_1 and m_2. If $m_1 < m_2$, we would expect that the dumbbell would oscillate just like a simple pendulum, around its present position. On the other hand, if $m_1 > m_2$, we would expect that the two-mass pendulum would swing downward until the masses settled into an inverted position, with m_2 at the top and m_1 at the bottom. Thus, this is a stability problem, with the operative question being: Is the configuration shown in Figure 7.3 a stable configuration?

Predict?

How?

To answer this question we must model the free vibration of the two-mass pendulum.We can build that model by extending the elementary pendulum model: First, we write the total energy for the dumbbell and then we differentiate that total energy to derive the equation of motion. Note that while there are two separate masses, only one degree of freedom, the angle, $\theta(t)$, is needed to specify the positions of both masses. Thus, taking our cue from eq. (7.20), the kinetic energy for the dumbbell is

$$KE_2 = \frac{1}{2}(m_1 + m_2)\left(l\frac{d\theta(t)}{dt}\right)^2. \qquad (7.60)$$

The potential energy of the swinging mass, PE, is measured with respect to a datum through the origin of the coordinates ($x = 0, y = 0$) in another familiar calculation:

$$PE_2 = m_1 g y_1(t) - m_2 g y_2(t) = -(m_1 - m_2) g l (1 - \cos\theta(t)). \quad (7.61)$$

For a linear two-mass pendulum model, we can approximate the potential energy as

$$PE_2 \cong -\frac{1}{2}(m_1 - m_2) g l (\theta(t))^2. \quad (7.62)$$

The total energy, $E_2(t)$, is found by adding eqs. (7.60) and (7.62):

$$E_2(t) = \frac{1}{2}(m_1 + m_2)\left(l\frac{d\theta(t)}{dt}\right)^2 - \frac{1}{2}(m_1 - m_2) g l\,(\theta(t))^2. \quad (7.63)$$

Then we can derive the equation of motion for the dumbbell by differentiating eq. (7.63) with respect to time,

$$\frac{dE_2(t)}{dt} = \left[(m_1 + m_2) l^2 \frac{d\theta^2(t)}{dt^2} - (m_1 - m_2) g l \theta(t)\right]\left(\frac{d\theta(t)}{dt}\right), \quad (7.64)$$

from which it follows that

$$(m_1 + m_2) l \frac{d\theta^2(t)}{dt^2} - (m_1 - m_2) g \theta(t) = 0,$$

or

$$\frac{d\theta^2(t)}{dt^2} + \frac{(m_2 - m_1)}{(m_1 + m_2)}\left(\frac{g}{l}\right)\theta(t) = 0. \quad (7.65)$$

Equation (7.65) is the same homogeneous, second-order, linear differential equation with constant coefficients that we solved before [i.e., eq. (7.42)] with the solution

$$\theta(t) = C e^{\lambda t}, \quad (7.66)$$

which leads to a characteristic equation for the constant, λ, that has two solutions,

$$\lambda_{1,2} = \pm j\sqrt{\frac{(m_2 - m_1)}{(m_1 + m_2)}\left(\frac{g}{l}\right)}. \quad (7.67)$$

Now the most interesting feature of eq. (7.67) is that the very nature of the roots,$\lambda_{1,2}$, changes according to the relative size of the two masses. For the case $m_2 > m_1$, the roots are purely imaginary, so the dumbbell will simply oscillate around its initial position (i.e., m_1 at the top and m_2 at the **Verified?**

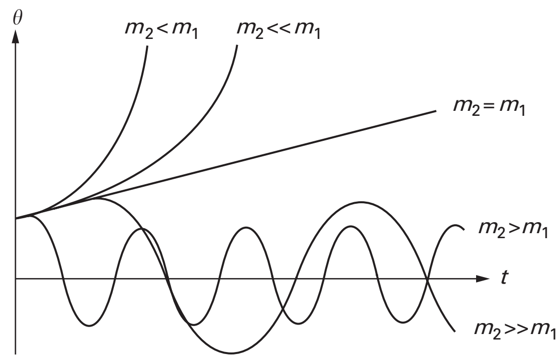

Figure 7.4 A sketch of the solutions to the linearized equations of motion of a dumbbell, a two-mass pendulum. These solutions are periodic when the initial configuration is stable ($m_2 > m_1$) and are exponential when the initial state is unstable ($m_2 < m_1$). The case $m_2 = m_1$ is a *critical point* that defines the border between the stable and unstable states.

bottom). On the other hand, if $m_2 < m_1$, the roots (7.67) become two *real* roots:

$$\lambda_{1,2} = \pm j\sqrt{\frac{-(m_1 - m_2)}{(m_1 + m_2)}\left(\frac{g}{l}\right)} = \mp\sqrt{\frac{(m_1 - m_2)}{(m_1 + m_2)}\left(\frac{g}{l}\right)}. \qquad (7.68)$$

Equation (7.68) mean that the two homogeneous solutions for $m_2 < m_1$ are exponentials, one decaying to zero, the other growing without bound. Thus, the case $m_2 < m_1$ represents an instance where the initial configuration is unstable, a finding that accords with our intuition of what would happen if we tried to stand a top-heavy dumbbell on its lighter end. Figure 7.4 shows a plot of schematic solutions for both real and imaginary values of the roots, for both the periodic and exponential solutions. The case $m_2 = m_1$ is a *critical point* that defines the border between a stable initial configuration ($m_2 > m_1$) and an unstable initial state ($m_2 < m_1$).

Thus, we have seen here an instance where changes in the parameters produce changes in the mathematical behavior of the model, which is a signal that different physical behavior is to be expected. An often-asked question in engineering and the physical sciences is whether a system's parameters support its bounded oscillation about its equilibrium position, or whether its instability is possible or even certain. We will see an instance of the former in a nonlinear biological model in Section 7.6.

Use?

7.5 The Freely-Vibrating Pendulum–III: The Nonlinear Model

We now return to the classical single pendulum to illustrate one of the most elegant solutions in applied mathematics and to show how an approximation to the nonlinear results can be obtained with some of the series introduced in Chapter 4. We begin with eq. (7.22) for the total energy of the pendulum, while also noting that the energy is a constant (eq. (7.24)) for this conservative system: **Why?** **How?**

$$\frac{1}{2}\left(\frac{d\theta(\tau)}{d\tau}\right)^2 + (1 - \cos\theta(\tau)) = \frac{E_0}{mgl}. \tag{7.69}$$

Now for a pendulum released from the resting position, $\theta(0) = \theta_0$, we can determine (see Problem 7.24) the constant, E_0, so that

$$\left(\frac{d\theta(\tau)}{d\tau}\right)^2 + 2(1 - \cos\theta(\tau)) = 2(1 - \cos\theta_0). \tag{7.70}$$

With the aid of a standard double-angle formula, we can rewrite eq. (7.70) as

$$\left(\frac{d\theta(\tau)}{d\tau}\right)^2 = 4\sin^2\frac{\theta_0}{2} - 4\sin^2\frac{\theta(\tau)}{2}. \tag{7.71}$$

We now introduce a constant,

$$p \equiv \sin\frac{\theta_0}{2}, \tag{7.72}$$

and a change of variable to a new angle, ϕ,

$$\sin\frac{\theta(\tau)}{2} \equiv \sin\frac{\theta_0}{2}\sin\phi = p\sin\phi, \tag{7.73}$$

so that the energy equation (7.71) can be written as

$$\left(\frac{d\theta(\tau)}{d\tau}\right)^2 = 4p^2\cos^2\phi. \tag{7.74}$$

Equation (7.74) does look neater and more elegant, but it has two dependent variables, θ and ϕ. However, we can differentiate eq. (7.73) to show that

$$\frac{1}{2}\cos\frac{\theta}{2}d\theta = p\cos\phi\, d\phi,$$

or

$$d\theta = 2p\frac{\cos\phi}{\cos\frac{\theta}{2}}d\phi = 2p\frac{\cos\phi}{\sqrt{1-p^2\sin^2\phi}}d\phi, \tag{7.75}$$

which allows us to rewrite eq. (7.74) as

$$d\tau = -\frac{d\phi}{\sqrt{1-p^2\sin^2\phi}}, \tag{7.76}$$

with a minus sign [for the square root of eq. (7.74)] that arises because $\theta(\tau)$ is measured positive counter-clockwise from the pendulum's perigee. Thus, for $\theta(0) = \theta_0 > 0$, we have both $d\theta/d\tau$ and $d\phi/d\tau < 0$.

Equation (7.76) can be formally integrated, but we must exercise care in choosing the limits. The period of the nonlinear model, $\widetilde{T}_0$, differs from the linear period, $T_0 = 2\pi/\omega_0$. In terms of the dimensionless time variable, $\tau = t\omega_0$, an integration over the first quarter of the period means that $0 \leq \tau \leq (\widetilde{T}_0\omega_0/4 = \pi\widetilde{T}_0/2T_0)$, and that $\pi/2 \leq \phi \leq 0$:

$$\frac{\widetilde{T}_0}{T_0} = -\frac{2}{\pi}\int_{\pi/2}^{0}\frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \frac{2}{\pi}\int_{0}^{\pi/2}\frac{d\phi}{\sqrt{1-p^2\sin^2\phi}}. \tag{7.77}$$

The integral on the right-hand side of eq. (7.77) is an *elliptic integral* (of the first kind), for which there are published tables of numerical values as a function of p. Thus, the tabulated values of the integral make it possible to calculate how the nonlinear period varies with p—which means how the nonlinear period, $\widetilde{T}_0$, varies with the initial amplitude of the pendulum, θ_0 (recall the definition of p in eq. 7.73)). This confirms what we said when we discussed the experimental data presented in Section 7.1.1: The period of oscillation of the pendulum does depend on its initial position or amplitude.

What happens with the linear model? The answer is that for very small values of θ_0, and thus of p, we make the same kind of approximation of the radicand in eq. (7.77) that we did in eqs. (7.37a–b): We say $1-p^2\sin^2\phi \cong 1$, in which case we recover the linear result, $\widetilde{T}_0 \cong T_0$.

The reduction to the linear case also suggests that we apply the binomial expansion (4.24) to the radicand in eq. (7.77) for small values of p:

$$\frac{\widetilde{T}_0}{T_0} = \frac{2}{\pi}\int_{0}^{\pi/2}\frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} \cong \frac{2}{\pi}\int_{0}^{\pi/2}(1+\frac{p^2}{2}\sin^2\phi)d\phi, \tag{7.78}$$

which, after integration and another application of the small-angle approximation, yields

$$\frac{\widetilde{T}_0}{T_0} \cong 1 + \frac{p^2}{4} = 1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} \cong 1 + \frac{\theta_0^2}{16}. \tag{7.79}$$

Once again we see here the dependence of the period on the amplitude, and the results predicted from eq. (7.79) can be compared both to the exact result given in eq. (7.77) and to the experimental data given in Table 7.1 (see Problems 7.25 and 7.26).

Problem 7.24. Determine the value of the constant energy, E_0, in eq. (7.69) for (a) a pendulum released from a resting position $\theta(0) = \theta_0$, and (b) for a pendulum given an initial speed $\dot{\theta}_0$ while hanging vertically ($\theta(0) = 0$).

Problem 7.25. Complete the integration of the last form of eq. (7.78) and confirm the first equality in eq. (7.79).

Problem 7.26. Use tabulated values of the elliptical integral of the first kind (eq. (7.77)) to determine the values of $\widetilde{T}_0/T_0$ for the values of θ_0 used in Table 7.1.

Problem 7.27. Compare and contrast the values found in the last column of Table 7.1 with the results found in Problem 7.26.

7.6 Modeling the Population Growth of Coupled Species

In Section 5.6 we introduced the logistic growth model that shows how, in a nonlinear fashion, the exponential growth of a single population or species can be bounded. What happens if there are *two* species that interact with each other? The Lotka-Volterra model of population growth provides an answer to this question, and in so doing it uses many of the modeling ideas developed above for the pendulum. The two-species model is of particular interest to biologists, with one species typically playing *host* to the second, *parasitic* population. **Why?**

The bounding effect of the single-population logistic model is produced by the inclusion of the term $-\lambda^2 N^2$ in the population balance equation **How?**

Assume? (see eqs. (5.48) and (5.50)). This term describes the *inhibition* of the population's growth. We start with two populations, the host (or prey) $H(t)$ and the parasite (or predator) $P(t)$, and we assume that the growth of *each* population is inhibited by the size of the *other* population. Thus, in the place of eq. (5.50) for a single population, we start with

$$\frac{dH(t)}{dt} = \lambda_{\mathrm{H}} H(t)\left(1 - \frac{P(t)}{P_{\mathrm{e}}}\right), \tag{7.80a}$$

and

$$\frac{dP(t)}{dt} = -\lambda_{\mathrm{P}} P(t)\left(1 - \frac{H(t)}{H_{\mathrm{e}}}\right). \tag{7.80b}$$

The positive constants, λ_{H} and λ_{P}, represent the uninhibited growth and decay rates, respectively, of the host and parasite populations, and each has physical dimensions of $(\text{time})^{-1}$. The population values, H_{e} and P_{e}, correspond to the equilibrium values of the two populations, the point at which the population rates, dH/dt and dP/dt, both vanish and the two populations are in *static* equilibrium with each other.

Verified? Equation (7.80a) shows that the parasite population reduces the growth rate of the host population, which is what parasites or predators do. On the other hand, the presence of the hosts in eq. (7.80b) slows the decline of the parasite population (for $H(t) < H_{\mathrm{e}}$), since there are fewer sources of sustenance when there are fewer hosts or prey. Thus, eqs. (7.80a–b)—which are variously known as the *Lotka-Volterra* equations or the *predator-prey* or *parasite-host* equations—do seem to be intuitively correct.

Further, while eqs. (7.80) resemble the single-population logistical model (5.50), there is one interesting and important distinction. While the single-population model (5.50) incorporated a maximum population $N_{\max}$, the predator-prey model refers to equilibrium populations that may be exceeded, which means that there could be a *change in the arithmetic signs* of the right-hand sides of eqs. (7.80a–b). For example, when $H(t) > H_{\mathrm{e}}$, the parasite decay rate turns into a growth rate. This suggests that the population sizes might *oscillate* or vibrate about their equilibrium sizes.

Equations (7.80) are *coupled*, *nonlinear*, ordinary differential equations. They are *coupled* because the dependent variables, $H(t)$ and $P(t)$, appear in both equations, and *nonlinear* because of the products of $H(t)$ and $P(t)$. No explicit solutions for $H(t)$ and $P(t)$ are known to exist for these nonlinear equations. However, as with the pendulum, we can use other means to extract a great deal of information.

7.6.1 Qualitative Solution for the Nonlinear Model

While we cannot explicitly integrate eqs. (7.80a–b), we can divide one by the other and obtain a form that is independent of the independent variable t: **How?**

$$\frac{dH}{dP} = -\frac{\lambda_H}{\lambda_P}\frac{(1 - P/P_e)H}{(1 - H/H_e)P}. \tag{7.81}$$

If the fractions in eq. (7.81) are cleared and the populations are rendered dimensionless with respect to their equilibrium populations, we find

$$\frac{1}{\lambda_H}\left(\frac{1}{H/H_e} - 1\right) d(H/H_e) + \frac{1}{\lambda_H}\left(\frac{1}{P/P_e} - 1\right) d(P/P_e) = 0. \tag{7.82}$$

Equation (7.82) can be straightforwardly integrated to yield

$$\frac{1}{\lambda_H}\left(\ln\frac{H}{H_e} - \frac{H}{H_e}\right) + \frac{1}{\lambda_P}\left(\ln\frac{P}{P_e} - \frac{P}{P_e}\right) = \text{constant}. \tag{7.83}$$

When plotted on the set of axes comprising the (H, P) space, eq. (7.83) represents a family of closed curves "centered" around the equilibrium point (H_e, P_e), as shown in Figure 7.5. Each member of the family of curves corresponds to a different value of the constant in eq. (7.83), with the area enclosed by the curve increasing with the value of the constant. We also

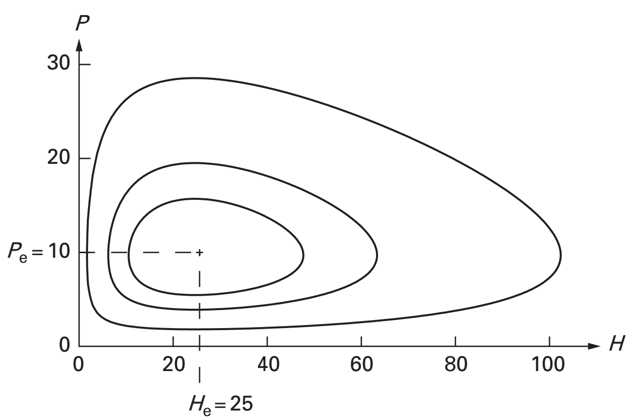

Figure 7.5 Three curves that illustrate the family of curves represented by eq. (7.83). Here $\lambda_H = 1.00$ per unit time, $\lambda_P = 0.50$ per unit time, $P_e = 10$ and $H_e = 25$. Note the equilibrium point, as well as the horizontal and vertical flat spots discussed previously, as well as the elliptical nature of the curves closest to the equilibrium point (Pielou, 1969).

note flat spots at abscissa values of $H = H_e$ that correspond to the vanishing of the slope dP/dH (or where $dP/dt = 0$ in eq. (7.80b)). Similarly, we note vertical tangents ("vertical flat spots") at ordinate values of $P = P_e$ that correspond to the slope dP/dH becoming infinite (or where $dH/dt = 0$ in eq. (7.80a)). More importantly, for given values of the constant, we can trace the magnitudes of the two populations and can thus examine how predator and prey or parasite and host interact.

7.6.2 Oscillatory Solution for the Linearized Model

How? A further examination of the curves in Figure 7.5 also shows that those nearest the equilibrium point are nearly elliptical in shape. Thus, let us write the values of $H(t)$ and $P(t)$ in the forms

$$\frac{H}{H_e} = 1 + \frac{h}{H_e} \quad \text{and} \quad \frac{P}{P_e} = 1 + \frac{p}{P_e}. \tag{7.84}$$

Assume? Let us further assume that the values of $h(t)$ and $p(t)$ are small compared to their respective equilibrium values of the populations:

$$\frac{h}{H_e} \ll 1 \quad \text{and} \quad \frac{p}{P_e} \ll 1. \tag{7.85}$$

Equations (7.84) and (7.85) provide a basis for generating binomial expansions of the natural logarithms in eq. (7.83). If that's done, the result is that to $O(h, p)^3$, eq. (7.83) becomes (see Problem 7.28):

$$\frac{1}{\lambda_H}\left(\frac{h}{H_e}\right)^2 + \frac{1}{\lambda_P}\left(\frac{p}{P_e}\right)^2 = \text{constant}. \tag{7.86}$$

Equation (7.86) is clearly that of an ellipse and so confirms the observation made above about the shapes of the closed curves near equilibrium.

What happens when we substitute eq. (7.84) into our original model equations (7.80a–b)? We would find that

$$\frac{dh(t)}{dt} = -\lambda_H H_e \left(1 + \frac{h(t)}{H_e}\right)\left(\frac{p(t)}{P_e}\right), \tag{7.87a}$$

and

$$\frac{dp(t)}{dt} = \lambda_P P_e \left(1 + \frac{p(t)}{P_e}\right)\left(\frac{h(t)}{H_e}\right). \tag{7.87b}$$

If we now linearize eqs. (7.87a–b) to keep only linear terms on their right-hand sides, we get

$$\frac{dh(t)}{dt} \cong -\lambda_H H_e \left(\frac{p(t)}{P_e}\right), \tag{7.88a}$$

and

$$\frac{dp(t)}{dt} \cong \lambda_P P_e \left(\frac{h(t)}{H_e} \right). \tag{7.88b}$$

We can now eliminate either of the functions $h(t)$ or $p(t)$ between eqs. (7.88a–b) to show that they each satisfy the same equation (see Problems 7.29 and 7.30):

$$\frac{d^2 h(t)}{dt^2} + \lambda_H \lambda_P h(t) = 0, \tag{7.89a}$$

and

$$\frac{d^2 p(t)}{dt^2} + \lambda_H \lambda_P p(t) = 0. \tag{7.89b}$$

Equations (7.89a–b) are the equations of simple harmonic oscillators! Thus, $h(t)$ or $p(t)$ represent small oscillations about the equilibrium position, a stable result. In fact, it is not hard to show (see Problems 7.31–7.33) that a solution to eqs. (7.88) or (7.89) is

$$\begin{aligned} p(t) &= p_0 \cos \sqrt{\lambda_H \lambda_P}\, t \\ h(t) &= -p_0 \sqrt{\frac{\lambda_H}{\lambda_P}} \left(\frac{H_e}{P_e} \right) \sin \sqrt{\lambda_H \lambda_P}\, t. \end{aligned} \tag{7.90}$$

where p_0 is a constant that will be determined by the initial conditions. In terms of the original host and parasite populations, the solution (7.90) appears as

$$\begin{aligned} P(t) &= P_e \left(1 + \frac{p_0}{P_e} \cos \sqrt{\lambda_H \lambda_P}\, t \right) \\ H(t) &= H_e \left(1 - \frac{p_0}{P_e} \sqrt{\frac{\lambda_H}{\lambda_P}} \sin \sqrt{\lambda_H \lambda_P}\, t \right). \end{aligned} \tag{7.91}$$

This result makes explicit the oscillation of the host and parasite populations around the equilibrium point. Moreover, the oscillations for both host and parasite occur at exactly the same natural frequency, $T_0 = 2\pi/\sqrt{\lambda_H \lambda_P}$. **Verified?**

It is worth noting that a potential instability phenomenon is embedded in the solutions (7.90) and (7.91). Recall that the uninhibited growth and decay rates, λ_H and λ_P, were assumed to be positive constants. If one of them were negative, that is, if the host population was declining or the parasite population growing, the outcome would be far different (see Problems 7.34 and 35). **Predict?** **Use?**

We close this discussion by noting that we have gained a great deal of information about host-parasite population systems without having obtained explicit solutions. We used both energy and small perturbation formulations to derive considerable *qualitative* understanding of the behavior of prey and predator. These qualitative approaches allowed us to identify the equilibrium point, the family of closed-curve solutions, the elliptical shapes of those curves in the neighborhood of equilibrium, and the periodic vibration of the two populations about equilibrium.

Problem 7.28. Use eqs. (7.84) and (7.85) to generate binomial expansions of the natural logarithms in eq. (7.83) and to confirm eq. (7.86) to $O(h, p)^3$.

Problem 7.29. Substitute $p(t)$ from eq. (7.88a) into eq. (7.88b) to obtain eq. (7.89a).

Problem 7.30. Substitute $h(t)$ from eq. (7.88b) into eq. (7.88a) to obtain eq. (7.89b).

Problem 7.31. Guided by the general solution (7.48), determine the solutions to eqs. (7.88) or (7.89) that satisfy initial conditions $p(0) = p_0$ and $dp(0)/dt = 0$.

Problem 7.32. What initial conditions are satisfied by $h(t)$ in the solution of Problem 7.31? Could they have been specified differently or separately?

Problem 7.33. What are the initial values of the populations $H(t)$ and $P(t)$ corresponding to the solution (7.90)?

Problem 7.34. What does it mean for the rate λ_P to become a negative constant?

Problem 7.35. Show how the solution (7.90) changes if λ_P is a negative constant.

7.7 Summary

In this chapter we have used the classical pendulum to show a mathematical model was derived, how it was grounded in and verified against experimental results, and how we could obtain qualitative information about its behavior. We also demonstrated the behavior of linear oscillators in several domains, and drew some distinctions between the behaviors exhibited by linear and nonlinear models. In so doing, we used concepts of linearity, dimensional consistency, scaling, and some basic ideas of second-order differential equations.

In terms of the behavior of the pendulum itself, we have shown how the period of the linear model depends only on the pendulum's properties and not on its amplitude of vibration, as is the case for nonlinear models wherein the amplitude is large. We also developed an elegant exact solution for the period of a pendulum and related it to the linear model. We also showed, for both the two-mass pendulum and a predator-prey population system, how the period of the vibrating system is sensitive to properties of that system—especially for the two-mass pendulum, for which instability occurs for certain combinations of masses.

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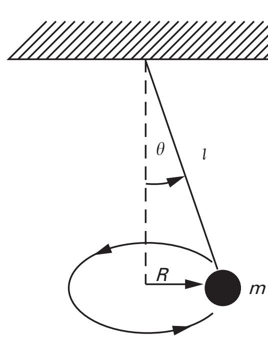
7.9 Problems

7.36. Use eq. (7.79) to determine the maximum angle, θ_0, such that the ratio, $\widetilde{T}_0/T_0$, does not exceed 1.005.

7.37. (a) Determine which variables affect the period of free vibration of the conical pendulum shown below from the accompanying table of data.

(b) Determine which variables affect the period of free vibration of the conical pendulum shown below using dimensional analysis.

		Period of Revolution (sec)					
θ	m	$l_1 = 1$ m			$l_2 = 3$ m		
θ_1	m_1	2.09	2.09	2.10	3.45	3.40	3.48
	m_2	2.07	2.08	2.08	3.46	3.44	3.44
θ_2	m_1	1.95	1.98	1.94	3.37	3.40	3.38
	m_2	1.96	1.93	1.95	3.36	3.38	3.35
θ_3	m_1	1.87	1.87	1.88	3.24	3.29	3.27
	m_2	1.86	1.85	1.87	3.22	3.25	3.21



7.38. Confirm the answer to Problem 7.37 (b) by deriving the equations of motion for a conical pendulum.

7.39. A uniform rod or stick is supported by and swings from a pivot at one end. The mass of this swinging rod is distributed over its length (unlike that of the classical pendulum introduced in Section 7.1). Use dimensional analysis to determine how the period of this pendulum depends on its mass per unit length, m, its length, l, and the gravitational constant, g.

7.40. Determine the period of the uniform rod or stick of Problem 7.39 by deriving its linearized (small angle) equation of motion. (*Hints*: Use Newton's laws of rotational motion, which then provide an analogy to the simple pendulum. The second moment of the rotational inertia is given as $I = ml^2/3$.)

7.41. Show that the total energy of the uniform rod or stick of Problem 7.40 is conserved. (*Hints*: The kinetic energy is given as $I(d\theta/dt)^2/2$. The potential energy is the pendulum's weight multiplied by the height of its mass center with respect to an appropriate datum.)

7.42. (a) Determine the rate at which energy is dissipated for a damped planar pendulum when the damping force is proportional to the square of the pendulum's speed.
(b) Confirm that the answer to part (a) is dimensionally correct.

7.43. (a) Write the equation for the total energy of an undamped linear spring-mass system in terms of its maximum displacement, A, and the spring stiffness, k.
(b) Confirm that the answer to part (a) is dimensionally correct.

7.44. Kepler's third law of planetary motion can be written as an equation for the square of a planet's period of motion around the sun,

$$T^2 = \frac{4\pi^2 a^3}{GM_s},$$

where a is the semi-major axis of the elliptical planetary orbit, M_s is the mass of the sun, and G is the universal gravitational constant. Further, Newton's first law states that the force of gravitation between the sun and a planet can be written as

$$F = \frac{GM_s(\text{mass of planet})}{(\text{distance from planet to sun})^2}.$$

(a) Starting with this form of Kepler's third law, find an equation for the frequency in the form $\omega = \omega(a, G, M_s)$.
(b) Determine the appropriate approximation of Newton's gravitational law to obtain Kepler's third law.

7.45. Explain whether or not energy is conserved in planetary motion. (*Hint*: The gravitational potential energy is GM_s(mass of planet)/(distance from planet to sun).)

7.46. Show from eq. (7.8b) that the mass of a simple pendulum attains its maximum speed when $\theta = 0°$. Is this physically reasonable?

7.47. Show that the result just obtained in Problem 7.46 is valid for both the linear and nonlinear models of the planar pendulum.

7.48. Would you expect to see energy conserved in laboratory experiments with pendulums? If not, how would the dissipation of energy make itself known?