5 Exponential Growth and Decay

This chapter is devoted to a discussion of *exponential models* that share a common characteristic: The rate of change of a variable, whether positive (as it grows) or negative (as it decays), is directly proportional to the immediate value of that variable. More often than not, the rate of change is a *time* rate of change that is proportional to the variable's *instantaneous* value. Similar exponential decays also occur spatially, that is, with respect to a spatial coordinate. Here, behaviors decay over some distance so as to have little or no effect at distances sufficiently far from the initiating behavior. We will see that exponential models are ubiquitous and have many applications, including in physics, finance, and population and resource predictions.

5.1 How Do Things Get So Out of Hand?

As we have just indicated, the primary characteristic of exponential growth or decay of a population is the dependence of the rate of growth of the population on its size at any instant. Thus, if a population is large, its growth rate will be proportionately large, and its continuing growth will accelerate with its increasing size. As we will soon see, this kind of growth exhibits itself in nice, smooth curves whose ordinate values increase very rapidly in relatively short periods of time. One application area where **Why?**

this behavior is often modeled is the field of population studies. Indeed, much has been made in recent years of the dangers of overpopulation and of the related resource and environmental issues. In fact, with regard to the principles of modeling outlined in Section 1.2, common sense would indicate in this instance that we have a pretty good idea of what we are looking for, what we know, and what we want to know.

Consider the two population projections shown in Figures 5.1 and 5.2. Even though they are now somewhat dated, both curves project very rapid increases in the world's population in relatively short times. The first curve (Figured 5.1) reflects both historical data for the years prior to 1960 and a projection from a 1960 world population estimate of 3 billion people growing at a rate of 2% per year. The world population was quite small until 1700, but it has been growing rapidly since the end of the 19th century.

Find? However, even though the projections past 1960 are at a modest rate of 2% per year, we should wonder about the validity of the steepness of the projection, especially after the year 2100.

If we were to extend the projection shown in Figure 5.1 for another 700 or 800 years, we would obtain the results shown in Figure 5.2. The assumed annual growth rate is still 2% and the population is still measured in billions. However, the time scale has been expanded by a factor of two and the population projections are now measured in *millions of billions*! While these population projections are almost certainly unrealistic, the

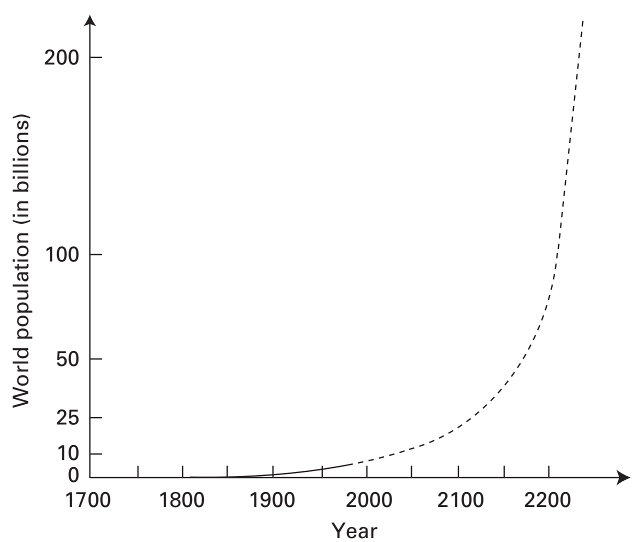

Figure 5.1 A historical view (solid line, for 1700–1960) and a projection (dashed line, for 1960–2165) of the world's population.

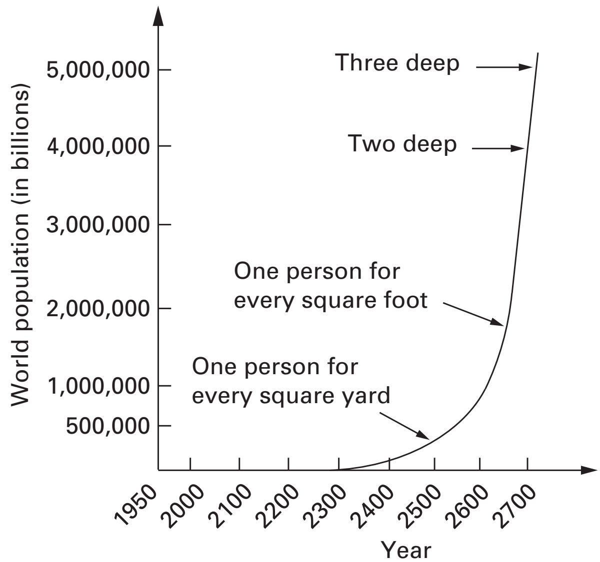

Figure 5.2 A longer projection, for 1950–2700, of the world's population, annotated to show the amount space that each person would be accorded were the projections to become reality.

projection curve clearly illustrates the nature of unrestrained exponential growth: The bigger it is, the faster it grows.

We also emphasize (again, as in Section 3.5.2) the importance of scaling when examining exponential growth. Consider the magnitudes of the numbers involved. For example, at a 2% annual growth rate, the world population grows from 3 billion in 1960 to 5,630,000 billion in 2692 (cf. Figure 5.2). What does it mean to have

5,630,000 billion people

or

5,630,000,000,000,000 people

or

$$5.63 \times 10^{15} \text{ people}$$

on earth in the year 2692? Is there room for all of these people? Could we even count this many people in a census? (And if you think that this is not a meaningful question or issue, there were vigorous debates within the U.S. Congress about the role of statistical sampling in the 2000 census—and they were talking about counting "only" some 285 million Americans!)

What? Let's try to answer the "room" issue first, that is, is there space enough on earth for more than five million billion people? The total surface area of the earth is approximately 5.10×10^8 km^2 ($\sim 5.49 \times 10^{15}$ ft^2), of which 72% **Given?** is water. Assuming that people cannot stand on water, the net "standing area" is approximately 1.43×10^8 km^2 ($\sim 1.54 \times 10^{15}$ ft^2). If each person were given just 1 square foot of personal standing space, people would have to be stacked more than three deep in order to accommodate everyone!

How long would it take to physically count all of the people on earth in **Assume?** 2692? Suppose we could tally the population at a rate of 1000 people per **Predict?** second. Then it would take

$$\frac{5.63 \times 10^{15} \text{ people}}{1000 \text{ people/s}} = 5.63 \times 10^{12} \text{ s}.$$

This seems like a lot of counting time. In fact, it easily shown that this simple calculation suggests that it would take almost 200,000 years to count the population growth that occurred in (only!) 800 years at a 2% annual growth rate.

We have presented the above numbers in part because they are patently **Valid?** absurd, to show just how things get out of hand. These numbers show how simplistic calculations with exponentials can lead to results that are arithmetically correct yet fail the test of basic credibility. We also note again the effect of scale in displaying such results. The ordinate scales of Figures 5.1 and 5.2 are linear and represent, respectively, 100 billion people per 1.50 in of graph and 2,000,000 billion per in. To express the projected population of 5.63×10^{15} people on the same ordinate scale of Figure 5.1, we would need a piece of paper that is 85,000 in long (you do the math!). It is also readily shown (see, for example, Problems 5.38 and 5.40) that exponential curves do not always portray such dramatic results.

Remember, therefore, that a change in scale does not, by itself, generate or dissipate true exponential behavior. Scale changes add or disguise **Improve?** perspective on the underlying mathematics. What is more important is that exponential behavior can express other kinds of response, illustrated in Figure 5.3, both of which occur when the proportionality factor is negative.Figure 5.3(a) shows a classic *decay* or dissipation curve in which an initial value decays to zero, while Figure 5.3(b) shows how some variable grows evermore slowly, *asymptotically*, to a limiting value as time becomes infinite. We will see both of these behaviors in Section 5.4, for example, when we describe the charging and discharging of a capacitor in a very elementary electrical circuit. Thus, after we introduce the necessary mathematics, we should also expect to see mathematical behavior that is

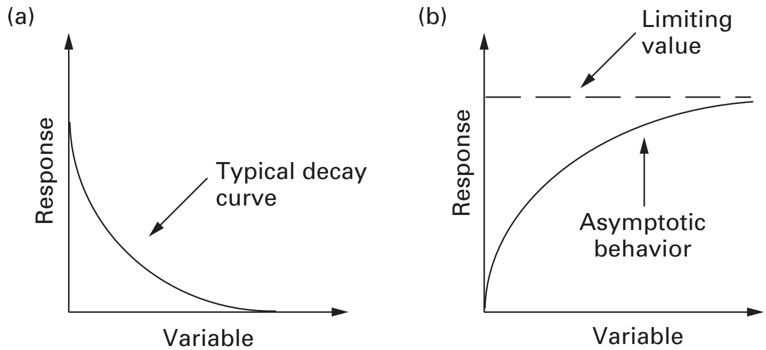

Figure 5.3 Illustrations of the kinds of exponential behavior that result when the constant of proportionality is less than zero (negative): (a) classic decay from a given initial value; and (b) asymptotic growth toward a limiting value or asymptote as time becomes indefinitely large.

more complicated and more interesting than simple, unrestrained exponential growth. We will then see that such exponential behavior is an important part of very practical and useful modeling in many disciplines. The foregoing discussion should, therefore, be taken as a cautionary "word to the wise" about some of the dangers in exponential modeling, not as a reason to dismiss or ignore it.

Problem 5.1. If you were asked to conduct a population study, what would you be looking for, what would be known, and what would you want to know?

Problem 5.2. What sort of assumptions would you make if you were asked to conduct a population study? On what basis could those assumptions be justified?

Problem 5.3. What factors might restrain or otherwise influence the unrestrained growth seen in Figures 5.1 and 5.2?

Problem 5.4. Determine the radius of the earth in both meters and feet from the surface areas given in Section 5.1. Are these values consistent with the conversion factors given in Table 2.3?

Problem 5.5. Confirm that it would take almost two hundred thousand years to count a population of 5.63 million billion people at a rate of 1000 people/s.

5.2 Exponential Functions and Their Differential Equations

In this section we first describe some of the arithmetic that underlies all of the numbers and curves given in Section 5.1. We follow that with a very brief primer on the first-order differential equation from whence derives the exponential function. This primer is intended to serve as a reminder of—not a substitute for—comparable introductory material in differential equations.

5.2.1 Calculating and Displaying Exponential Functions

The exponential behavior discussed in Section 5.1 can be put in mathematical terms as follows. Let $N(t)$ be the number or population of a collection of objects, and let t be the independent variable on which N depends and with which it changes. For most of our applications, t will be associated with time, but that is a result of the models we exhibit, not due to any underlying mathematical requirement. As we indicated in Section 5.1, exponential growth results when *the rate of growth is proportional to a population or number.* If we introduce a *constant of proportionality*, λ, then exponential growth occurs when

$$\frac{dN(t)}{dt} = \lambda N(t). \tag{5.1}$$

We see from eq. (5.1) that the constant of proportionality λ can be written as

$$\lambda = \frac{dN/N}{dt}. \tag{5.2}$$

Thus, λ represents the fractional change dN/N of the population per unit change of the independent variable, dt. The dimensions of λ are seen to be

$$[\lambda] = \frac{1}{[t]} = [t^{-1}]. \tag{5.3}$$

If the independent variable, t, is a measure of time, then the dimensions of λ are 1/time.

Equation (5.1) is a first-order differential equation that is linear in the dependent variable $N(t)$ and has constant coefficients. As we show in the next section, eq. (5.1) has a solution that can be written as

$$N(t) = N_0 e^{\lambda t}, \tag{5.4}$$

where the constant N_0 is an *arbitrary constant* whose value remains to be determined. The dimensions (and units) of N_0 must be the same as those of $N(t)$. Further, the number e is the *base of the natural logarithm.* It has an approximate value $e \cong 2.71828$. Since $e^0 = 1$, it also follows that the number N_0 must be the *initial value* of the population, that is, the number of objects whose change we are modeling at $t = 0$, when the model "starts." Note, too, that $N(t)$ grows in time if λ is positive, much like the curves in Figures 5.1 and 5.2, and that it decreases in magnitude or decays if $\lambda < 0$, as does Figure 5.3(a).

Since e is the base of natural logarithms, we can take the (natural) logarithm of both sides of eq. (5.4) to show that

$$\lambda t = \ln N(t) - \ln N_0 = \ln \left(N(t)/N_0\right). \tag{5.5}$$

Equation (5.5) tells us that if we want to find a time, t_n, when the number $N(t_n) = nN_0$, that is, when the population size is a specified multiple of its initial value, all we need to do is calculate

$$t_n = \frac{\ln n}{\lambda}. \tag{5.6}$$

People frequently ask how long it takes something to double in size, in which case the answer is the *doubling time*, t_2, determined from eq. (5.6) with $n = 2$:

$$t_2 = \frac{\ln 2}{\lambda} \cong \frac{0.693}{\lambda}. \tag{5.7}$$

One immediate application of eq. (5.7) is to investment: Money grows as it earns interest. Suppose that we want to know how long it would take to double an amount of money with continuously compounded interest. We determine that by interpreting λ in terms of percentage, P, in which case $P = 100\lambda$. Then eq. (5.7) becomes **Why?** **Find?**

$$t_2 = \frac{69.3}{P}. \tag{5.8}$$

The approximate time it would take to double some money as a function of different percentage growth rates P is shown in Table 5.1.

There are two other interesting properties of exponential growth. The first is the *inversion* of the doubling time that occurs when we calculate the *half-life* of a population. That is, suppose we want to know how long it takes for a population that started at N_0 to decrease to a value of $N_0/2$. In this case, λ would represent a (negative) decay rate, and from eq. (5.6) we would get a formula for the half life $t_{1/2}$ that is formally identical to eq. (5.7) or eq. (5.8). Thus, we need only change the column headings in Table 5.1 to "Annual Decay ($P < 0$, %)" and "Half-Life ($t_{1/2}$, years)," respectively, to obtain the variation of half-life as a function of decay rate.

Table 5.1 The time it takes to double one's money, measured in years, as a function of continuously compounded growth rates, measured in percentages.

Annual Growth (P, %)	Approximate Doubling Time (t_2, years)
1	69.3
2	34.6
5	13.9
10	6.93
20	3.46

The second interesting property is this. The time, t_n, it takes for a population, $N(t)$, to grow by a constant factor, n, remains unchanged throughout the growth. Thus, from time $t = 0$ to $t = t_2$, the population doubles; from $t = t_2$ to $t = 2t_2$, the population doubles again; and so on. Thus, we obtain the results shown in Table 5.2.

Finally, for this section, some remarks on the display of exponential functions are now in order. We saw in Section 5.1 that exponential growth can lead to some horrifically large numbers. However, in the same way that great strengths and great weaknesses are often intertwined, it is similarly the case that the logarithms of exponential growth provide the means of graphical (and representational) salvation. If we look back at eq. (5.5), we see that one representation of exponential behavior can be expressed in the form:

$$\ln N(t) = \lambda t + \ln N_0. \tag{5.9}$$

Equation (5.7) suggests that a *semi-logarithmic plot* of ln $N(t)$ against λt (plus a constant) would produce results in which the ordinate values are

Table 5.2 The growth of the exponential function as gauged by multiples of the doubling time.

Time (units of t_2)	Population ($N(t)$)	
$t = 0$	$N = N_0$	$= 2^0 N_0$
$= t_2$	$= 2N_0$	$= 2^1 N_0$
$= 2t_2$	$= 4N_0$	$= 2^2 N_0$
$= 3t_2$	$= 8N_0$	$= 2^3 N_0$
$= 10t_2$	$= 1024N_0$	$= 2^{10} N_0$
$= nt_2$		$= 2^n N_0$

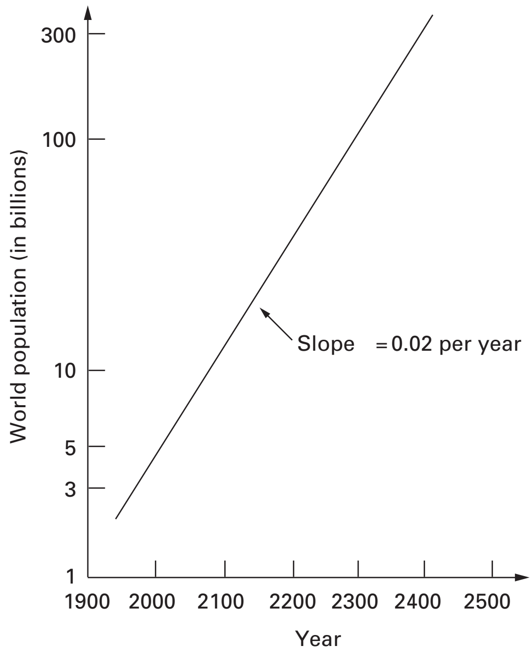

Figure 5.4 Population projections for the period 1960–2400 based on the data of Figures 5.1 and 5.2, presented herein in a *semi-logarithmic* plot. What had previously been displayed as a set of steeply rising exponential curves is now seen as a relatively benign straight line with ordinate values in particular that are much more manageable.

more commensurate with those of the abscissa. In fact, in such a semi-logarithmic plot, eq. (5.9) represents a straight line of slope λ and with intercept $\ln N_0$. In Figure 5.4 we show such a linear "semi-log" using the projected data of Figures 5.1 and 5.2.

Problem 5.6. Confirm by differentiating eq. (5.4) that the $N(t)$ given therein satisfies eq. (5.1).

Problem 5.7. How would the projections of Figures 5.1 and 5.2 change if the growth rate were, respectively, 1% per year and 3% per year?

Problem 5.8. What annual growth rate would be needed to double one's money in seven years?

5.2.2 The First-Order Differential Equation $dN/dt - \lambda N = 0$

There is another interesting property of exponential behavior that has been present but to which we have not paid much attention in our discussion thus far. This special property is the fact that there is only *one* arbitrary constant in the basic exponential model [see the discussion immediately after eq. (5.4)]. Why is that so? There is only one constant because, as we will now demonstrate, the exponential function (5.4) is the solution to a first-order differential equation, that is, a differential equation in which the highest-order derivative is of first order. The single arbitrary constant arises from the fact that a first-order differential equation needs to be integrated just once to obtain a solution.

Consider the differential equation governing population growth set out in eq. (5.1):

$$\frac{dN(t)}{dt} - \lambda N(t) = 0. \tag{5.10}$$

This differential equation has *constant coefficients*, that is, the multipliers of both $N(t)$ and its derivative are constants, namely, λ and 1, respectively. Equation (5.10) can also be written in the form

$$\frac{dN(t)}{N(t)} - \lambda dt = 0, \tag{5.11}$$

which can be integrated in exactly the same way that the Naperian logarithm was defined in Section 4.9 and then inverted to yield the solution (see Problem 5.9):

$$N(t) = Ce^{\lambda t}. \tag{5.12}$$

We can clearly identify C as the initial population by setting $t = 0$ in eq. (5.12). Equally clearly, we can identify that initial value in the notation introduced in eq. (5.4): $C = N_0$.

The initial value C need not be determined at the time $t = 0$. We could specify a starting condition that at some time t_0, $N(t_0) = N_0$. Equation (5.12) then dictates that

$$N(t_0) = N_0 = Ce^{\lambda t_0},$$

which means in turn that

$$C = N_0 e^{-\lambda t_0}. \tag{5.13}$$

If we substitute this form of our constant of integration C into the solution (5.12), we get

$$N(t) = N_0 e^{-\lambda t_0} e^{\lambda t} = N_0 e^{\lambda(t-t_0)}. \tag{5.14}$$

This obviously defines a population that for $\lambda > 0$ is increasing through N_0 at $t = t_0$, but that is less than N_0 for $t < t_0$.

Note that all of the foregoing manipulations are as valid for $\lambda < 0$ as they are for $\lambda > 0$. The interpretations would obviously be different, since we would be describing exponential decay ($\lambda < 0$) rather than exponential growth ($\lambda > 0$), but the underlying mathematics is unchanged. However, it is also true that the analysis to date is limited by the fact that our basic differential equation (5.10) is a *homogeneous equation*, that is, there is no *forcing function* on the right-hand side. When we discuss the charging of a capacitor in a simple electrical circuit in Section 5.4, we will see that the charge $q(t)$ in the capacitor in that circuit is described by an equation of the form

$$\frac{dq(t)}{dt} + \frac{1}{RC}q(t) = V_{in}(t). \tag{5.15}$$

Equation (5.15) looks very much like the differential equation (5.10) for exponentials, except that it has a *forcing function*, $V_{in}(t)$, on the right-hand side that forces or drives the change of the voltage in the circuit being modeled. Further, the coefficient in eq. (5.15) is equivalent to taking $\lambda = -(1/RC) < 0$ in eq. (5.10).

Problem 5.9. Verify that eq. (5.12) is the solution to the exponential differential equation as given in eq. (5.10) by using the result that

$$\int \frac{du}{u} = \ln u + \text{constant}.$$

Problem 5.10. Show that the solution (5.12) to the differential equation (5.10) can also be found by assuming the following trial solution for $N(t)$:

$$N(t) = Ce^{\alpha t}.$$

Problem 5.11. Why is the proportionality constant in eq. (5.15) equivalent to having $\lambda < 0$ in eq. (5.10)? What sort of behavior would we then expect?

5.3 Radioactive Decay

We now want to model the decay of radioactive isotopes as exponential behavior. As physicists and chemists began to study radioactivity at the **Why?**

end of the 19th century, they found that the activity of radioactive isotopes decreased with time at rates that varied with the material. When the emission of α and (primary) β particles was observed in the laboratory, it was found that the number of particles collected over time was unaffected by changes in pressure, temperature, chemical state, or the physical environment. Instead, the observed *half-life* of each isotope—the time it takes for the number of particles of the isotope to be reduced by half—was found to be a characteristic of the material itself. Thus, once half-life is identified as a material property, a measurement of a radioactive decay pattern can be used to identify a material by its characteristic half-life.

Find?

Given?

In Figure 5.5 we show a generic, semi-logarithmic plot of the radioactive decay of an unspecified material. It strongly resembles Figure 5.4. In the radioactive decay model, however, the proportionality constant is negative (i.e., $\lambda < 0$). Further, in Figure 5.5, we have rendered the abscissa dimensionless by measuring it in terms of an (unknown) half-life, $t_{1/2}$. That is,

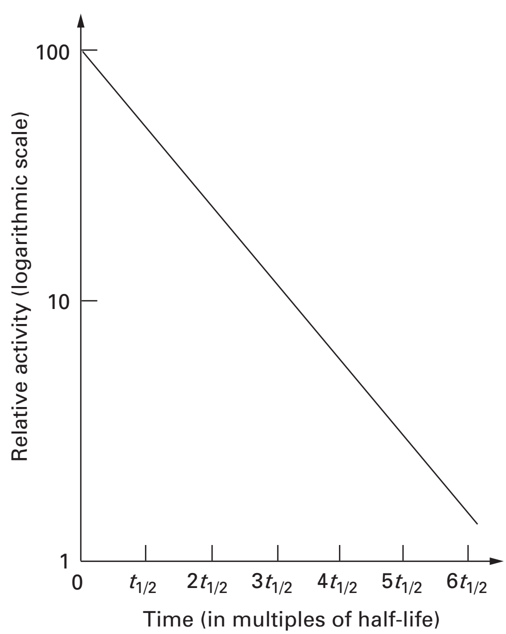

Figure 5.5 A generic plot of the decay of a radioactive isotope. Note that the data is presented in a *semi-logarithmic* plot. Note, too, that the abscissa has been made dimensionless by measuring it in terms of an (unknown) half-life, $t_{1/2}$.

time is measured here as a multiple of a parameter, $t_{1/2}$, whose dimensions (of time) are given but whose specific numerical value is not.

Decay rates are often used to characterize emitters as *short-lived* or *long-lived*. For example, consider the decay of the element thorium (Th in the atomic table of the elements). Thorium has a half-life of 16,500,000,000 yr, which does seem like quite a long time! If that is indeed true, can we calculate the effective decay constant, λ, and estimate how many thorium atoms will decay in a year? **Predict?**

We can calculate λ by applying eq. (5.6) with $n = 1/2$ and $t_{1/2} = 1.65 \times 10^{10}$ yr. Then the decay constant can be calculated as **How?**

$$\begin{aligned}\lambda &= \frac{-0.693}{t_{1/2}} = \frac{-0.693}{1.65 \times 10^{10}\,\text{yr}} = -4.20 \times 10^{-11}\,\text{yr}^{-1} \\ &= -4.20 \times 10^{-11}\frac{1}{\text{yr}} \times \frac{\text{yr}}{365\,\text{day}} \times \frac{\text{day}}{86{,}400\,\text{sec}} \\ &= -1.33 \times 10^{-18}\,\text{sec}^{-1},\end{aligned} \tag{5.16}$$

where *reciprocal seconds* are the units ordinarily used to express radioactive decay constants. In view of the definition (5.2) of decay rate in terms of fractional population change, eq. (5.16) suggests that only one thorium atom in every $(1.33 \times 10^{-18})^{-1} = 7.51 \times 10^{17}$ such atoms decays in one second. Indeed, even in a year, only one of every 2.38×10^{10} thorium atoms present initially will decay. Thus, it does seem that thorium can be safely characterized as a long-lived emitter.

It is worth touching on two related points here. One is that the characterization of a radioactive emitter as short- or long-lived seems, in the above context, a straightforward and neutral piece of scientific reasoning. However, similar calculations done in other contexts (e.g., the decay time for radioactive waste in a national storage facility for radioactive materials from nuclear power plants, or the remediation time for gasses to fully dissipate from a landfill) often turn these characterizations into political (and emotional) debates that try to define the meaning of "short (or long) enough for ..." **Use?**

The second point is a deeply philosophical one about the very underpinnings of the models of physics. What does it mean for a fraction of a single isotope or atom to decay? Or, are the models really about averages calculated over a large number of particles? And, if that is the case, how are such averages calculated? And, further, what is the meaning of the various levels of models that are used to describe and predict these behaviors? **Valid?** **Improve?**

5.4 Charging and Discharging a Capacitor

Why? We will now model the behavior of a very simple electrical circuit, incorporating only two *passive* electrical elements, a *capacitor* defined by its capacitance, C, and a *resistor* defined by its resistance, R. These two elements are depicted in Figure 5.6. **Assume?** The first step in our circuit modeling is to identify a functional relationship for each element, called a *constitutive equation*, which expresses its behavior in terms of the voltage drop across the element and the current flowing through it.

How? The capacitor stores and discharges energy. This energy transfer occurs as charge is transferred from one side plate or electrode to the other (viz., Figure 5.6(a)) and, in this process produces a voltage drop across the capacitor given by:

$$[V_a(t) - V_b(t)]_C \equiv \Delta V_C(t) = \frac{q(t)}{C}, \tag{5.17}$$

where $\Delta V_C(t)$ represents the voltage drop across the capacitor while the *charge*, $q(t)$, flows through it. In SI units, C, the *capacitance*, is measured in coulombs (of charge) per volt or farads.

Keep in mind that while we are used to talking about current flowing through electrical devices in everyday life, here we are building our model in terms of the charge, $q(t)$, whose first derivative in time is the *current*,

$$i(t) \equiv \frac{dq(t)}{dt}. \tag{5.18}$$

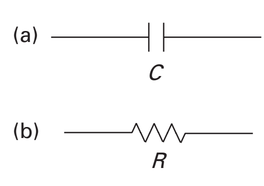

Figure 5.6 Simple conceptual drawings of the *icons* or symbols of two passive electrical elements: (a) the *capacitor*, denoted by C, stores energy by storing charge and discharges energy through the flow of charge (or current, the time rate of change of charge); and (b) the *resistor*, R, that allows charge or current to flow, but that in so doing dissipates some of the energy flow as wasted thermal energy.

We will return to and amplify this point in Section 8.7 wherein we model circuits more extensively and relate the electrical elements to analogous mechanical elements.

The second passive element, the resistor depicted in Figure 5.6(b), impedes or resists the flow of charge (or current) as the charge flows through the element. The resistor thus dissipates energy, usually perceived as wasted heat. The voltage drop across a resistor is usually expressed in terms of voltage and current as *Ohm's law*:

$$[V_a(t) - V_b(t)]_R \equiv \Delta V_R(t) = Ri(t), \tag{5.19}$$

where $\Delta V_R(t)$ represents the voltage drop across the resistor while the current, $i(t)$, flows through it. In SI units, R, the *resistance*, is measured in volts per ampere or ohms. Since we are interested in expressing our current model in terms of charge, we make use of eq. (5.18) to eliminate the current from eq. (5.19) and rewrite Ohm's law as:

$$\Delta V_R(t) = R\frac{dq(t)}{dt}. \tag{5.20}$$

5.4.1 A Capacitor Discharges

Having modeled our two circuit elements, we now model the simple electrical circuit shown in Figure 5.7. That picture shows a resistor in series with a capacitor, and with an (externally) applied voltage across the circuit's two "free" endpoints or nodes. Suppose first that no voltage is applied across the free endpoints. In that case, it seems quite logical to stipulate that the sum of the voltage drops across the capacitor and the resistor must simply vanish because nothing is being put into the system, that is,

$$\Delta V_C(t) + \Delta V_R(t) = 0. \tag{5.21}$$

If we substitute eqs. (5.17) and (5.20) into eq. (5.21), we can replace its voltage terms and express it entirely in terms of the charge $q(t)$ flowing around this simple circuit:

$$\frac{dq(t)}{dt} + \frac{1}{RC}q(t) = 0. \tag{5.22}$$

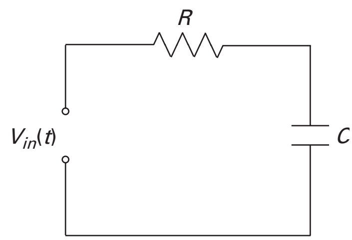

Figure 5.7 A very simple electrical circuit that connects a *capacitor*, C, in series with a *resistor*, R, and an externally applied input voltage, $V_{in}(t)$. Here we have directly connected the two elements rather than showing their individual nodes, but we have shown the two "free" nodes or endpoints that normally would serve as the terminals to which we would attach a battery or some other voltage supply.

Predict? The resemblance between eqs. (5.22) and (5.10) is unmistakable, so it follows immediately that the solution to eq. (5.22) can be written as [see eq. (5.12)]

$$q(t) = C_1 e^{-t/RC}, \tag{5.23}$$

where C_1 is an arbitrary constant that can be taken as the initial charge: $C_1 = q_0 = q(t = 0)$. Equation (5.23) shows that the capacitor's initial charge, q_0, left on its own, theoretically vanishes as $t \to \infty$. (In practice, the initial charge becomes so small that we can say it has vanished.) Described in Section 5.1, this behavior was shown in Figure 5.3(a).

Valid? We also see from eq. (5.23) that the behavior of a simple *RC circuit* occurs in times that we can express and measure in terms of a characteristic constant, namely, *RC*. This means, first of all, that the decay of the charge is inversely proportional to both the resistance and the capacitance. It is intuitively satisfying to see that the decay will be slowed if either the capacitor is large, in which case it can hold a larger charge that will take longer to dissipate, or if the resistance is large, in which case the discharge of current through the resistor will be slowed down. Second, it is not surprising that one widely used measure of the decay rate of such a circuit is a *time constant*, τ, defined as:

$$\tau = RC. \tag{5.24}$$

The time constant, τ, is the time it takes for an initial charge, q_0, to be reduced to the value, q_0/e. With the definition (5.24), the RC circuit's governing equation can be written as

$$\frac{dq(t)}{dt} + \frac{1}{\tau}q(t) = 0. \tag{5.25}$$

Note also that with the governing equation written this way, dimensional consistency is much easier to discern and to verify.

Problem 5.12. Verify that each term in eq. (5.22) has the same physical dimensions.

Problem 5.13. Confirm that eq. (5.23) is the correct solution to eq. (5.22).

Problem 5.14. Use the definitions of resistance and capacitance to verify that the product RC has the physical dimensions of time.

5.4.2 A Capacitor Is Charged

Find? Can we extend the foregoing circuit model to charge the capacitor? We can, by inserting a voltage source across the two free endpoints of the RC circuit in Figure 5.7. (We should not confuse this with the familiar experience of charging a dead car battery by connecting it with jumper cables to a good battery because that charging results from a relatively rapid conversion of electrical energy to chemical energy.) How do we incorporate a voltage source to revise our circuit model?

How? There are two (at least) ways to answer this question. First, we would extend the reasoning behind eq. (5.21) by simply adding to that equation a term representing the input voltage $V_{in}(t)$ supplied by a battery or an equivalent device:

$$\Delta V_C(t) + \Delta V_R(t) = V_{in}(t). \tag{5.26}$$

Then, with the appropriate constitutive equations and the definition of the circuit's time constant, eq. (5.26) can be rewritten as

$$\frac{dq(t)}{dt} + \frac{1}{\tau}q(t) = \frac{V_{in}(t)}{R}. \tag{5.27}$$

The differential equation (5.27) is called *inhomogeneous* in the terms of mathematics because the voltage input makes its right-hand side take on a non-zero value.

Valid? We can also demonstrate that eq. (5.27) is a correct model by applying a classical result of electrical circuit analysis, *Kirchhoff's Voltage Law* (KVL), named after the German physicist Gustav Robert Kirchhoff (1824–1887). Kirchhoff observed that *the algebraic sum of the voltage drops across all of the elements connected in a closed circuit loop is zero.* Written in symbolic terms, the KVL looks like the following:

$$\sum_{k=1}^{K} [V_a(t) - V_b(t)]_k = \sum_{k=1}^{K} \Delta V_k(t) = 0, \tag{5.28}$$

where K is the total number of elements in the closed circuit loop. Note that we must pay close attention to the sign conventions built into the constitutive laws of the circuit elements when we apply the KVL because it calls for calculating an "algebraic sum" of the voltage drops. The KVL can be applied to the circuit in Figure 5.7 (see Problem 5.16) to find once again the result in eq. (5.27).

To return to our stated modeling task of charging a capacitor, let us apply eq. (5.27) under the simple assumption of a constant input voltage, $V_{in}(t) = V_0 = \text{constant}$:

$$\frac{dq(t)}{dt} + \frac{1}{\tau} q(t) = \frac{V_0}{R}. \tag{5.29}$$

Remembering that the derivative of a constant is zero, it is not very hard to show (see Problem 5.17) that we can construct a solution to eq. (5.29) in the form

$$q(t) = V_0 C + C_1 e^{-t/\tau} = \tau V_0/R + C_1 e^{-t/\tau}. \tag{5.30}$$

Once again the single arbitrary constant, C_1, can be determined from the circuit's given initial conditions. In the simpler case where the initial charge is supplied only by the voltage input, it follows from eq. (5.30) that

$$q(0) = 0 = V_0 C + C_1.$$

The arbitrary constant is now determined and the complete correct solution becomes:

$$q(t) = V_0 C (1 - e^{-t/\tau}). \tag{5.31}$$

Predict? Equation (5.33) is plotted in Figure 5.8, which is a more detailed version of the sketch given in Figure 5.3(b). We see that the charge increases exponentially from its initially given value of zero. Here, however, the amount of

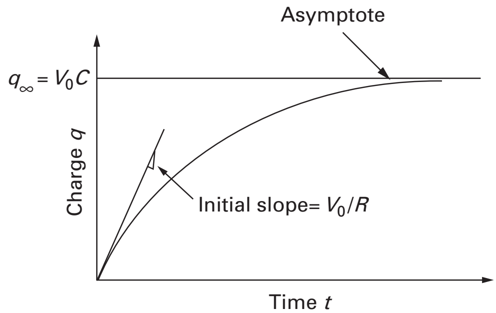

Figure 5.8 The charge in the capacitor when an input voltage, V_0, is applied across the terminals shown in Figure 5.7. Note the unmistakable resemblance of this drawing to the sketch in Figure 5.3(b). Here the initial slope, V_0/R, and the asymptotic value of the charge, $q_\infty = V_0C$, are called out.

charge does not increase to infinity. Instead, it *asymptotically* approaches a maximum value given by

$$q_\infty \equiv q(t \to \infty) = V_0C. \tag{5.32}$$

This asymptotic value of the charge, $q_\infty = V_0C$, is the maximum value that the capacitor can hold for the given capacitance, C, and applied voltage, V_0. We also can calculate that the initial slope of the charging curve is V_0/R. If this slope were zero, then no charging would be possible and the charge would remain at its initial value of zero. Of course, this circumstance could only arise if no voltage were applied or if the resistance to that applied voltage was infinitely large. Thus, once again we have found results that are intuitively pleasing.

One last point here. We have charged a capacitor even though the circuit's **Improve?**
decay constant is negative, that is, $\lambda = -1/\tau$. We have thus imposed growth on an exponential system that otherwise would have decayed. This serves to point out that external conditions, such as the input voltage applied here, can influence the behavior of an exponential system to an extent not anticipated by the sign of the constant λ.

Problem 5.15. Verify that eq. (5.29) was properly derived from eq. (5.28).

Problem 5.16. Apply the KVL of eq. (5.28) to the circuit in Figure 5.7 to confirm the validity of eq. (5.27).

Problem 5.17. Confirm by differentiating eq. (5.31) that it satisfies eq. (5.29).

Problem 5.18. Calculate the initial slope of the charging capacitor from the solution given in eq. (5.31).

5.5 Exponential Models in Money Matters

Why? We will now talk about money, that is, we will model elementary exponential behavior as seen in two important aspects of our financial lives. First, we will talk about *interest* and *compound interest*, the repeated calculation of interest over shorter periods of time that produces higher effective rates of interest than may be evident. Then we will describe *inflation*, the phenomenon we see when prices rise rapidly and dramatically.

5.5.1 Compound Interest

It is hard to listen to the news these days without hearing reports on the stock and bond markets. When the markets and their underlying economies are not doing well, we hear about whether or not the Federal Reserve Bank will adjust the interest rates that the banks, including "The Fed," charge each other on interbank loans. We are besieged by advertisements promising high interest returns on various savings instruments and low interest rates on credit card balances and mortgage loans. For all this talk of interest, however, few understand that interest is an exponential phenomenon, which is one reason that economists speak of the *time value of money*, and that very serious consequences follow inattention to interest rates and compounding practices.

Find? Predict? Consider first the latest unsolicited offer of a credit card promising an interest rate of only 0.75% per month, which is advertised as "only" 9% per year and sounds cheap in this context. If the monthly interest was compounded on a monthly basis, the effective annual interest rate is found from the 12-fold multiplication

$$(1.0075)(1.0075)\cdots(1.0075) = (1.0075)^{12} \cong 1.0938. \tag{5.33}$$

Thus, monthly compounding produces an effective annual interest rate of about 9.38% per year. If these rates were *continuously compounded*, we would use eq. (5.4) to find:

$$\frac{N(t)}{N_0} = e^{(0.0075)(12\text{mos})} \cong 1.0942, \tag{5.34}$$

which represents an effective rate of 9.42% per year. Thus, depending on how interest is applied or compounded, the *effective interest rate* charged on a 9% (nominal) card would be 9.39% for monthly compounding and 9.42% for continuous compounding. (United States law requires that advertisements and transaction documents list the nominal, uncompounded APR or *Annual Percentage Rate*, with compounding details and effective rates often left to the fine print.) If these effective interest rates don't seem like a very big deal, consider that they add noticeable surcharges to the nominal rates.

We can also see the effects of compounding by looking at returns on investment. Suppose that interest is promised at a nominal rate of 10% per year. That interest could be calculated and distributed in discrete amounts of 10% annually, 5% semiannually, 2.5% quarterly, and so on. For m compounding periods per year, the initial investment would grow to: **How?**

$$\left(\frac{N}{N_0}\right)_m = \left(1 + \frac{0.10}{m}\right)^m. \tag{5.35}$$

We have shown some results for various compounding intervals in Table 5.3. Note that the investment promises larger returns as the number of compounding periods, m, is increased. Thus, it seems interesting to consider what will happen to the value of the unit investment as the number of compounding periods becomes infinitely large. **Find?**

Table 5.3 The growth of a unit investment (i.e., $N_0 = 1$) at a nominal rate of 10% with returns compounded and payable m times per year. Equation (5.35) is used to calculate that growth.

Number of Compounding Periods per Year (m)	Value of a Unit Investment ($N_0 = 1$)
0	1.0000
1	1.1000
2	1.1025
4	1.1038
12	1.1047
365	1.1051559

We can easily answer this question by first recasting eq. (5.35) in terms of a new variable $x = m/0.10$. Then eq. (5.35) becomes: **How?**

$$\left(\frac{N}{N_0}\right)_m = \left(1 + \frac{1}{x}\right)^{0.10x}. \tag{5.36}$$

We now take the limit of eq. (5.36) as x becomes infinitely large:

$$\left(\frac{N}{N_0}\right)_\infty = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^{0.10}. \tag{5.37}$$

Within this limit lies, in fact, the formal definition of the base e of the natural logarithm:

$$e \equiv \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x. \tag{5.38}$$

Find? We thus see that our unit investment, continuously compounded, attains in one year a value only slightly larger than the daily compounding shown in the last line of Table 5.3:

$$\left(\frac{N}{N_0}\right)_\infty = e^{0.10} \cong 1.1051709. \tag{5.39}$$

Use? It is worth noting that, for economists, interest represents the price of money. What does that mean? Putting money into a savings account means giving up an opportunity to buy something *now* in exchange for the promise of being able to spend a larger amount of money—the initial investment plus earned interest income—*at a future date*. This means trading the opportunity to spend \$1.00 now for the opportunity to spend \$1.10 a year from now. The bank has "purchased" money for its own investment purposes at a price of \$0.10 for the year, and the saver bought the chance to spend still more money, \$1.10, one year later. This means that money has both a price and, again, a *time value* because investors make decisions about what their money will be worth in the future. This brings us to a second money issue, inflation, in which exponential behavior significantly affects the price of money.

Problem 5.19. Are eq. (5.35) and (5.38) related? How?

Problem 5.20. Verify all of the steps that lead from eq. (5.35) to eq. (5.39).

Problem 5.21. Construct a version of Table 5.3 for an annual interest rate of 18%.

5.5.2 Inflation

Why? Inflation has been a major economic and political problem in the United States at various times in the 20th century, and it has troubled and even destabilized the economies of many other countries in just the last few years. Asian economies suffered major bouts of inflation in the late 1990s, and at the very end of 2001 Argentina had street riots and five (!) presidents in less than two weeks because of economic problems triggered in part by serious inflation. Inflation occurs when the value of money declines and the prices of goods and services rise accordingly. Countries suffering from bouts of inflation see the value of their currencies drop against those of other countries, and the consequences of such economic imbalances may include unemployment, trade embargoes and trade wars, and severe, spreading economic dislocation. These topics are the province of economics, "the dismal science," but they are interesting to us because inflation is an exponential phenomenon and the mathematics of inflation is provocative.

Find?

Consider first simple price inflation as measured by the purchase price of gasoline. Gasoline cost a nickel a gallon in 1933, while at the end of 2001 it cost \$1.00 per gallon. We can calculate the annual price inflation rate for gasoline with eq. (5.5):

$$\lambda_{price} = \frac{\ln(1.00/0.05)}{68\,\text{yr}} \cong 0.0440/\text{yr}, \tag{5.40}$$

Predict?

which corresponds to a price inflation rate of 4.40% per year. This price inflation rate caused gasoline's price at the pump to go up by a factor of 20 in 68 years.

Valid?

Improve?

As appealing as this simple calculation may be, it would be quite misleading to say that the *real* price of gasoline went up twentyfold during the time 1933–2001 because, while the *nominal* or apparent price of gasoline was going up, the value of the dollar itself was going down. That is, inflation affects not only the price of goods and services; it also affects the price of money. During the 68 years included in the previous calculation, the value of the dollar declined substantially, because a 1933-dollar and a 2001-dollar are only *nominally* the same. If we assume that the dollar was losing its purchasing power at only 2% per year, we could calculate the value of a single dollar after t years, $v(t)$, from eq. (5.14) with $\lambda_\$ = 0.02/\text{yr}$:

$$\frac{v(t)}{v(1933)} = e^{-\lambda_\$(t-1933)} = e^{-0.02(t-1933)}. \tag{5.41}$$

Thus, after 1, 10, and 68 years, the purchasing power or value of a 1933-dollar would be \$0.98, \$0.82, and \$0.26, respectively. So, after almost 70 years, the 2001-dollar has turned out to be worth little more than one-quarter of the 1933-dollar!

Use?

However, an economist would view this differently. Recall from Section 5.5.1 that we noted that interest is the price of money bought in a forward-looking transaction.Thus, we can rephrase the question about the loss of value in the dollar into a purchasing question: How much would one have to pay in 1933 to have \$1.00 available in 2001? That is a price question answered simply by inverting eq. (5.41):

$$\frac{v(1933)}{v(t)} = e^{+\lambda_{\$}(t-1933)} = e^{+0.02(t-1933)}. \tag{5.42}$$

So, repeating the calculation just done in this different form, a purchaser would have to invest \$0.98, \$0.82, and \$0.26, respectively, in order to have \$1.00 available to spend in the years 1934, 1943, and 2001. Equation (5.42) thus can be said to represent the currency inflation rate.

Predict? Purchases can then be assessed either in terms of their current sales prices or in terms of *inflation-adjusted dollars* that support the calculation of a real economic price that reflects changes in a currency's purchasing power. We would calculate that *real* price by subtracting the currency inflation rate from the price inflation rate, that is,

$$\lambda_{real} = \lambda_{price} - \lambda_{\$}. \tag{5.43}$$

Equation (5.43) then states that the real inflation rate over the time interval 1933–2001 is then, from the example data given above, $\lambda_{real} = 4.40 - 2.00 = 2.40\%$.

We do not mean to suggest that inflation is an easy problem because it can be modeled with exponential mathematics. The foregoing analyses have truly simplified the world of economics. Economics has become in recent times a mathematically-oriented social science, as evidenced in part by the sophisticated mathematical models that led to the prizes won by most recent Nobel laureates. However, we do want to point out that the **Improve?** cumulative effects of percentages in economics can be enormous. We have ignored some measures that have been developed to deal with inflation, such as *indexing*, in which intended benefits are linked to a cost or price index, such as the oft-cited CPI, the *consumer price index*. We have also completely ignored the effects of technical innovation, productivity changes, new sources of energy, and many other factors that affect prices. Suffice it to say that the economics and politics of exponential growth in monetary affairs merit attention.

Problem 5.22. If gasoline cost \$0.70/gallon in 1978, calculate and compare the price inflation rates for the intervals 1933–1978 and 1978–2001.

Problem 5.23. If the cost of money exceeds the cost of goods, what happens to λ_{real}?

Problem 5.24. Speculate on the potential effects of λ_{real} staying negative for long periods of time.

5.6 A Nonlinear Model of Population Growth

In Sections 5.1 and 5.2 we discussed population growth and projections based on an elementary exponential model in which the population growth rate is *linearly* or directly proportional to the current size of the population. While we focused exclusively on growth rates, we could extend such linear models to account for mortality or death rates simply by taking the growth rate in eq. (5.1) as an effective or *net* rate that reflects the difference between birth and death rates: **Improve?**

$$\lambda_{effective} = \lambda_{birth} - \lambda_{death}. \tag{5.44}$$

In fact, we could also account for immigration and emigration in the analysis of the population changes of a particular country by writing a balance law much like eqs. (1.1) and (1.2) and accounting for the various growth and decay rates as: **How?**

$$\frac{dN(t)}{dt} = (\lambda_{birth} - \lambda_{death} + \lambda_{immigration} - \lambda_{emigration})N(t). \tag{5.45}$$

However, it is certain that these models either grow or decay monotonically, as simple exponentials, no matter how much we refine the details of these linear growth and decay rates. The fundamental behavior is unchanged, so that if we find the classic model inadequate, we need to change that model in a different way.

We would like to expand the notion of exponential growth to incorporate the idea of *limited* growth. There are many factors that do limit growth and that modelers have tried to incorporate into population projections, including resources, both renewable and nonrenewable, energy, capital (money), food supplies and distribution mechanisms, education, and family planning. These models were very popular in the late 1970s, but they were also both complicated and, by some, derided as unrealistic. Much of the complexity of those models stemmed from the fact that many of the growth variables are coupled, that is, the amount of capital formulation may depend on pollution indices and on energy availability, as well as on the instantaneous supply of money. Further, the right-hand side of eq. (5.1) may be more complex because the relationships among single or coupled variables may not be linear. How could that be? **Why?**

It could be more complex if we were to think of the right-hand side of eq. (5.1) as a Taylor series of a nonlinear function of $N(t)$ that is not yet defined. Thus, we would start by replacing eq. (5.1) by a more general formulation **Assume?**

$$\frac{dN(t)}{dt} = f(N(t)), \tag{5.46}$$

which states that the rate of growth of a population $N(t)$ is equal to some undefined function of the population, $f(N(t))$. Then, as we did for series representations of functions in Chapter 4, we could expand that function into a Taylor series such that:

How?

$$\frac{dN(t)}{dt} = f(N(t)) = C_0 + C_1 N(t) + C_2 N^2(t) + \cdots . \tag{5.47}$$

We would have to say first that $C_0 = 0$ simply because the growth rate of a population should be zero whenever the population size is zero. The constant C_1 must be our traditional growth rate, say λ_1. Then there are other constants, C_i, to evaluate, depending on how many terms we choose to keep in this series representation of $f(N(t))$. How do we evaluate these other constants?

Predict?

We illustrate that by narrowing our focus to a particular quadratic approximation in which eq. (5.47) takes the form:

$$\frac{dN(t)}{dt} = \lambda_1 N(t) - \lambda_2 N^2(t), \tag{5.48}$$

wherein both of the parameters λ_1 and λ_2 are taken as positive: $\lambda_1, \lambda_2 > 0$. In eq. (5.48) λ_1 corresponds to the population's uninhibited or net growth rate. The meaning of λ_2 emerges from noting that the rate of growth vanishes when $N(t) = N_{\text{max}}$:

$$\lambda_1 N_{\text{max}} - \lambda_2 N_{\text{max}}^2 = 0,$$

or when

$$\frac{1}{\lambda_2} = \frac{N_{\text{max}}}{\lambda_1}. \tag{5.49}$$

Thus, the reciprocal of λ_2 is the time needed for the maximum obtainable population to be achieved by uninhibited growth. On the other hand, with the aid of eq. (5.49), we can eliminate λ_2 from that the nonlinear equation and write it as:

$$\frac{dN(t)}{dt} = \lambda_1 N(t) \left(1 - \frac{N(t)}{N_{\text{max}}}\right). \tag{5.50}$$

Equation (5.50) shows a modification of the elementary exponential model where the growth rate is reduced by a factor representing the proportion of *unrealized population growth*, that is, the population represented by the difference between the maximum and instantaneous population values:

$$\frac{dN(t)}{dt} = \lambda_1 N(t) \left(\frac{N_{\text{max}} - N(t)}{N_{\text{max}}}\right). \tag{5.51}$$

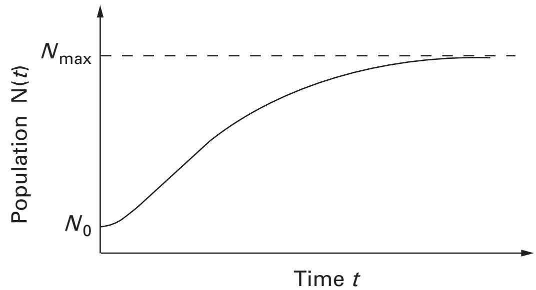

Figure 5.9 The *logistic growth curve*, shown as a model of limited or bounded growth as the population $N(t)$ moves from an initial population value, N_0, to its maximum value, N_{max}. It is plotted for the values $N_0 = 1$, $N_{max} = 10$, and $\lambda_1 = 1$.

There is a closed-form solution to eq. (5.51), despite the nonlinearity, and it is (see Problem 5.28):

$$N(t) = \frac{N_{max}}{1 + \left(\dfrac{N_{max} - N_0}{N_0}\right) e^{-\lambda_1 t}}, \tag{5.52}$$

where $N(t = 0) = N_0$ is the initial population. We have plotted eq. (5.52), known as the *logistic growth curve*, in Figure 5.9. Note that we can recover both the initial value of the population at $t = 0$, as well as the maximum value as time becomes indefinitely large.

We should observe again that we have not exhausted by any means the spectrum of exponential growth models. Nevertheless, we have shown here that models can lead to restricted or limited growth, which should provide some interest in exploring different exponential models in greater detail.

Problem 5.25. Look up the U.S. birth, death, immigration and emigration figures for the 10 decades of the 20th century and use the balance equation (5.45) to calculate the population changes that these rates predict.

Problem 5.26. How do the predictions of Problem 5.25 compare with the actual U.S. census data?

Problem 5.27. What are the implications for the model of eq. (5.48) of loosening the restriction that $\lambda_1, \lambda_2 > 0$?

Problem 5.28. Confirm by differentiating eq. (5.52) that it satisfies eq. (5.51).

5.7 A Coupled Model of Fighting Armies

We will now examine another exponential model wherein the complication of coupled populations is addressed. This model and the resulting *Lanchester's law* are named after Frederick William Lanchester (1868–1946), a remarkable British aeronautical engineer who wrote serious works on economics and fiscal policy, the theory of relativity, military strategy, as well as aerodynamics. **Why?** Lanchester wanted to describe the attrition of opposing forces at war. **Find?** Following this attrition required modeling the changes of two army populations whose respective rates of attrition depend on the size of the opposing army. **Given?** Thus, there are two armies whose attrition or decay rates are of interest, each of whose decay rates are proportional to the size of the other force. We will identify the two army populations as friendly forces, $F(t)$, and enemy forces, $E(t)$. Since the rate of change of $F(t)$ depends on $E(t)$ and vice versa, we say that these variables are *coupled*, or that we are solving a *coupled problem*. This model also has the nice feature, encapsulated in Lanchester's law, that we can obtain a great deal of information with a *qualitative* approach to the governing differential equations. We will use qualitative analyses to describe energy conservation and dissipation for a vibrating pendulum in Sections 7.1.5 and 7.1.6 and for the interaction of predators and prey in Section 7.6.

Consider that at some time, t, we have populations $F(t)$ of friendly troops and $E(t)$ of enemy troops. **Assume?** Further, as we intended, let us assume that the rates of change of their respective populations are proportional to the opposing combat force's size:

$$\begin{aligned}\frac{dF(t)}{dt} &= -\lambda_E E(t),\\ \frac{dE(t)}{dt} &= -\lambda_F F(t).\end{aligned} \tag{5.53}$$

How? The parameters, λ_E and λ_F, respectively, represent the effectiveness of the enemy and friendly forces, with interesting units: λ_E is the loss rate per unit time of friendly troops *per enemy troop*. Thus, if λ_E is larger than λ_F, the enemy troops are more effective because more friendly troops are lost per unit time *and* per unit of enemy forces.

Use? Equation (5.53) also shows more explicitly the meaning of coupling in a set of equations. Simply put, dF/dt depends on $E(t)$, and dE/dt depends on $F(t)$. That is why the pair of eqs. (5.53) are called *coupled equations*. It can be shown that this coupled pair of first-order equations is equivalent to a single second-order equation by, for example, simply treating the first of eq. (5.53) as an equation that defines $E(t)$, and then substituting it into the second of eq. (5.53):

$$\frac{dE(t)}{dt} = -\frac{1}{\lambda_E}\frac{dF^2(t)}{dt^2} = -\lambda_F F(t), \tag{5.54a}$$

which is easily rearranged into the form:

$$\frac{dF^2(t)}{dt^2} - \lambda_E\lambda_F F(t) = 0. \tag{5.54b}$$

Once this uncoupled second-order equation is solved for its single dependent variable, $F(t)$, the second dependent variable, $E(t)$, can be found without further integration (see Problem 5.29).

The formal solution to eqs. (5.53) or (5.54) can be found in terms of hyperbolic sines and cosines, which are also exponential functions. We will not do that here, although the form of eq. (5.54b) should be recalled when we discuss the vibration of pendulums in Chapter 7. Instead, we will show here how we can obtain a lot of information without formally solving the governing equations. We do this by first multiplying the first of eq. (5.53) by $\lambda_F F(t)$ and the second of eq. (5.53) by $\lambda_E E(t)$, after which we find: **How?** **Predict?**

$$\begin{aligned} \lambda_F F(t)\frac{dF(t)}{dt} &= -\lambda_E\lambda_F F(t)E(t), \\ \lambda_E E(t)\frac{dE(t)}{dt} &= -\lambda_F\lambda_E E(t)F(t). \end{aligned} \tag{5.55}$$

Since the right-hand sides of eq. (5.55) are the same, it follows that:

$$\lambda_F F(t)\frac{dF(t)}{dt} = \lambda_E E(t)\frac{dE(t)}{dt}. \tag{5.56}$$

It is easy to show that eq. (5.56) is equivalent to the statement that:

$$\frac{d}{dt}(\lambda_F F^2(t) - \lambda_E E^2(t)) = 0,$$

or

$$\lambda_F F^2(t) - \lambda_E E^2(t) = \text{constant}. \tag{5.57}$$

The constant in eq. (5.57) must have the same value it had at the beginning of the combat being modeled. With $E_0 = E(t = 0)$ and $F_0 = F(t = 0)$, it follows that:

$$\lambda_F F^2(t) - \lambda_E E^2(t) = \lambda_F F_0^2(t) - \lambda_E E_0^2(t),$$

or

$$\lambda_F(F^2(t) - F_0^2(t)) = \lambda_E(E^2(t) - E_0^2(t)). \tag{5.58}$$

Use? Equation (5.58) is called *Lanchester's square law*. We can use the square law to calculate the final size of the winning army when the enemy forces have been annihilated *without solving any differential equations*. Assume that victory is declared when all of the enemy forces are gone from the scene. In this case, $E_{final} = 0$. The number of friendly troops remaining then follows from eq. (5.58) as:

$$F_{final}^2(t) = F_0^2(t) - \frac{\lambda_E}{\lambda_F} E_0^2(t). \tag{5.59}$$

Thus, even in victory the number of (surviving) friendly troops is reduced by an amount proportional to the square of the initial size of the enemy force.

Valid? The dependence of the friendly and enemy force sizes on the respective squares has intriguing consequences. Suppose that two equally effective armies oppose each other. This means $\lambda_E = \lambda_F$, and that Lanchester's law [eq. (5.58)] becomes:

$$F^2(t) - E^2(t) = F_0^2(t) - E_0^2(t). \tag{5.60}$$

Suppose further that two combat scenarios were being considered by military planners. In the first scenario, a friendly army of 50,000 soldiers faces an enemy force of 40,000 and then meets a second enemy force of 30,000 soldiers. In the second scenario, the same friendly army meets an enemy force of 70,000, that is, it meets the same number of enemy troops assembled for a single fight. In the sequential scenario, the friendly army prevails in the first of its two battles with a surviving forces of 30,000 because $[(50{,}000)^2 - (40{,}000)^2] = (30{,}000)^2$, which is just enough to force a draw with the enemy in the second battle. If the armies meet in the second scenario, however, the friendly forces lose by a significant margin because $(50{,}000)^2$ is less than $(70{,}000)^2$. This clearly shows that strategy is important, especially that well-known precept of *divide and conquer*!

Improve? Verified? Of course, all of the Lanchester results are predicated on the rate equations (5.53), an assumption that must be kept in mind when the model is exercised. Suitably modified to include other effects (e.g., introducing reinforcements), the Lanchester model has modeled the outcomes of famous battles such as Iwo Jima (see Problems 5.47 and 5.48).

Problem 5.29. Assuming that eq. (5.54b) can be solved for $F(t)$, show that $E(t)$ can be determined without further integration.

Problem 5.30. Confirm that eq. (5.57) does follow from eq. (5.55) via eq. (5.56).

Problem 5.31. Suppose you are given the solution for the enemy population that satisfies eq. (5.53) as

$$E(t) = E_0 \cosh \alpha t - \sqrt{\frac{\lambda_F}{\lambda_E}} F_0 \sinh \alpha t,$$

where $\alpha^2 = \lambda_E \lambda_F$. How much time does it take for the enemy forces to be completely annihilated?

Problem 5.32. Would the strategy of divide and conquer work for a "linear attrition law" that for equally effective armies replaces eq. (5.57) with

$$F(t) - E(t) = F_0(t) - E_0(t)?$$

5.8 Summary

We dealt with a wide variety of exponential behavior models in this chapter, including population growth, radioactive decay, charging and discharging of capacitors, inflation and interest, and armies at war. While some of the behavior was about decay, it is the cases of exponential growth that really draw our attention. We saw the importance of scale in presenting and assessing various growth phenomena. We noted that decay effects can be modified by external inputs, such as the charge in a capacitor responding to an applied voltage. We also explored the nonlinear logistic growth model and the coupled Lanchester square law.

It is worth noting that we have touched on some very timely issues. At the same time, we have not "solved" any of these very real "problems." But we have shown that the models chosen can influence our projections and perceptions of these problems, as well as the ways we might approach them in the "real world".

5.9 References

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5.10 Problems

5.33. Show that if it takes time, t_c, to count a population, $P(t)$, that has a growth rate of λ, the population will increase by an amount equal to $\lambda t_c P(t)$.

5.34. (a) If the population counting rate is c, how long does it take to count the population at time, t?

(b) How much time does it take to count the increase in population that occurred while it was being counted at time t?

5.35. Find the actual world population figures for 1970, 1980, 1990, and 2000. Use these data to update the projections shown in Figures 5.1 and 5.2.

5.36. Using the 2% growth rate, plot the world population from 1960 to 2060 with an ordinate scale of 10 billion people per 1.50 in. Does this curve look like "reasonable" exponential growth?

5.37. The ordinate scales of Figures 5.1 and 5.2 are, respectively, 1.5 in = 100 billion people and 1.5 in = 3 million billion people. How much paper is needed to plot the 1960 world population of 3 billion people and the projected 2692 world population of 5.63×10^{15} people using each of those scales?

5.38. Plot the growth of world population from a 1960 value of 3 billion people at growth rates of 1, 2, and 3% per year through 2700 using semi-logarithmic paper. What shapes are these curves? What are their slopes and intercepts?

5.39. What is the time constant, comparable to that for an RC circuit, for a population decaying at a rate per unit time λ?

5.40. How much should be set aside in 2002 in a savings account earning 5.5% per year to accumulate $1,000,000 by 2022? By 2042?

5.41. Suppose that there was a steady inflation rate of 3% per year, what would the investments of Problem 5.40 have to be to accumulate $1,000,000 in 2042 measured in 2002 dollars?

5.42. The noted (and recently deceased) historian Stephen Ambrose has chronicled the growth of American railroads by listing the following amounts of total track by decade: 726 mi (1834), 4311 (1844), 15,675 (1854), and 33,860 (1864). Determine:
(a) the decade-by-decade growth rate; and
(b) the growth rate for exponential growth across all the data given.

5.43. Verify by differentiation and substitution that the following solution satisfies eqs. (5.53):

$$F(t) = F_0 \cosh \alpha t - \sqrt{\frac{\lambda_E}{\lambda_F}} E_0 \sinh \alpha t,$$

$$E(t) = E_0 \cosh \alpha t - \sqrt{\frac{\lambda_F}{\lambda_E}} F_0 \sinh \alpha t.$$

5.44. Confirm that the solution verified in Problem 5.43 satisfies Lanchester's square law of eq. (5.58).

5.45. The initial strengths of two opposing armies are $F_0 = 10{,}000$ and $E_0 = 5000$ troops, with equal loss rates of 0.1 per day. Who will win? How long will the battle take? (*Hint*: See Problem 5.31.) How many troops will the victor have when the enemy is vanquished? Graph the army populations until the enemy is completely annihilated.

5.46. The initial strengths of two opposing armies are $F_0 = 10{,}000$ and $E_0 = 5000$ troops, and $\lambda_F = 0.1$ per day. Who will win and with what remaining forces if $\lambda_E = 0.2$, 0.5, and 1.0 per day? What value of λ_E would produce a draw?

5.47. The landmark World War II battle of Iwo Jima began with troop sizes of $F_0 = 54{,}000$ and $E_0 = 21{,}500$ troops, with $\lambda_F = 0.0106$ per day and $\lambda_E = 0.0544$ per day. Absent any reinforcements, how long would this battle have lasted? How many troops would the victor have when the loser's forces were totally exhausted?

5.48. In order to end the fight for Iwo Jima in 28 days, how many troops would the United States have had to have initially? How do the U.S. losses in this scenario compare to those found in the scenario of Problem 5.47?