3
Scale

In this chapter we will continue dealing with dimensions, but focusing now on issues of *scale*, that is, issues of *relative size*. Size, whether absolute or relative, is very important because it affects both the form and the function of those objects or systems being modeled. Scaling inßuencesÑ indeed, often controlsÑ the way objects interact with their environments, whether we are talking about objects in nature, the design of experiments, or the representation of data by smooth, nice-looking curves. We even Þnd references to scaling in literature, such as in the depiction by satirist Jonathan Swift of the treatment accorded the traveler Gulliver when he arrived in the land of Lilliput:

His Majesty's Ministers, finding that Gulliver's stature exceeded theirs in the proportion of twelve to one, concluded from the similarity of their bodies that his must contain at least 1728 of theirs, and must needs be rationed accordingly.

This chapter is devoted to explaining where the factor of 1728 came from, as well as discussing abstraction and scale, size and shape, size and function, scaling and conditions that are imposed at an objectÕs boundaries, and some of the consequences of choosing scales in both theory and experimental measurements.

3.1 Abstraction and Scale

We start with some thoughts about the process of deciding on the appropriate level of detail for whatever problem is at hand, which also means deciding on the appropriate level of detail for the corresponding model. We call this process *abstraction*. It typically requires an organized, thoughtful approach to identifying those phenomena to which we really want to pay attention. In addition, thinking about *scaling* often requires that we think in terms of the magnitude or size of quantities measured with respect to a standard that has the same physical dimensions.

For example, a linear elastic spring can be used to model more than just the relation between force and relative extension of a simple coiled spring, as in an old-fashioned butcherÕs scale or an automobile spring. We could, for example, use $F = kx$ to describe the static load-deßection behavior of a diving board, but the spring constant k should reßect the stiffness of the diving board taken as a whole, which in turn reßects more detailed properties of the board, including the material of which it is made and its own dimensions. The validity of using a linear spring to model the board can be ascertained by measuring and plotting the deßection of the boardÕs tip as it changes with standing divers of different weight.

We had noted in Section 1.3.1 that the classic spring equation is also used to model the static and dynamic behavior of tall buildings as they respond to wind loading and to earthquakes. These examples suggest that we can use a simple, highly abstracted model of a building by aggregating various details within the parameters of that model. That is, the stiffness k for a building would incorporate or lump together a great deal of information about how the building is framed, its geometry, its materials, and so on. For both a diving board and a tall building, we would need detailed expressions of how their respective stiffnesses depended on their respective properties. We could not do a detailed design of either the board or of the building without such expressions. Similarly, using springs to model atomic bonds means that their spring constants must be related to atomic interaction forces, atomic distances, sub-atomic particle dimensions, and so on.

Another facet of the abstraction process is that in each case we are saying that, for some well-deÞned purposes, a Òreal,Ó three-dimensional object behaves like a simple spring. We are thus introducing the concept of a *lumped element* model wherein the actual physical properties of some real object or device are aggregated or *lumped* into a less detailed, more abstract expression. An airplane, for example, can be modeled in very different ways, depending on our modeling goals. To lay out a ßight plan or trajectory, the airplane can simply be considered as a point mass moving with respect to a spherical coordinate system. The mass of the point can simply be taken as the total mass of the plane, and the effect of the surrounding atmosphere can also be modeled by expressing the retarding drag force as acting on the mass point itself with a magnitude related to the relative speed at which the mass is moving. If we want to model and analyze the more immediate, more local effects of the movement of air over the planeÕs wings, we would build a model that accounts for the wingÕs surface area and is complex enough to incorporate the aerodynamics that occur in different ßight regimes. If we want to model and design the ßaps used to control the planeÕs ascent and descent, we would develop a model that includes a system for controlling the ßaps and also accounts for the dynamics of the wingÕs strength and vibration response.

Clearly, as we talk about Þnding the right level of abstraction or the right level of detail, we are simultaneously talking about Þnding the right *scale* for the model we are developing. *Scaling* or imposing a scale includes assessing the effects of geometry on scale, the relationship of function to scale, and the role of size in determining limits. We must think about all of these ideas when we are determining how to scale a model in relation to the ÒrealityÓ we want to capture.

Lastly, we often look at the scale of things with respect to a magnitude set within a standard. Thus, when talking about freezing phenomena, we expect to reference temperatures near the freezing point of materials included in our model. Similarly, we know that the models of Newtonian mechanics work extraordinarily well for virtually all of our earth- and space-bound applications. Why is that so? Simply because the speeds involved in all of these calculations are far, far smaller than c, the speed of light in a vacuum. Thus, even a rocket Þred at escape speeds of 45,000 km/hr seems to stand still when its speed is compared to $c \approx 300{,}000\,\text{km/s} = 1.080 \times 10^9\,\text{km/hr}$! These scaling ideas also represent something of an extension of the ideas behind dimensionless variables that we discussed in Chapter 2. For example, in EinsteinÕs general theory of relativity, the mass of a particle moving at speed, v, is given as a (dimensionless) fraction of the rest mass, m_0, by

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-(v/c)^2}}. \tag{3.1}$$

The scaling issue involved here, as we will discuss in Section 3.4, is ensuring that the square of the dimensionless speed ratio is always much less than 1, so that $m \cong m_0$.

3.2 Size and Shape: Geometric Scaling

In Figure 3.1 we show two cubes, one of which has sides of unit length in any system of units we care to choose, that it, the cubeÕs volume could be 1 in^3 or 1 m^3 or 1 km^3. The other cube has sides of length L in the same system of units, so its volume is either L^3 in^3 or L^3 m^3 or L^3 km^3. Thus, for comparisonÕs sake, we can ignore the units in which the two cubesÕ sides

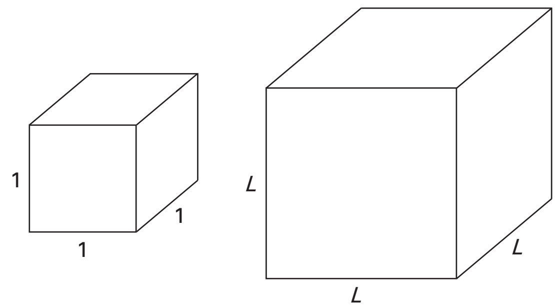

Figure 3.1 Two geometrically similar cubes, one with sides of *unit length* (that is, having lengths equal to 1 measured in any system of units), and the second with sides of length L as measured in the same units as the "unit cube."

are actually measured. The total area and volume of the Þrst cube are, respectively, 6 and 1, while the corresponding values for the second cube are $6L^2$ and L^3. We see immediately an instance of *geometric scaling*, that is, the area of the second cube changes as does L^2 and its volume scales as L^3. Thus, doubling the side of a cube increases its surface area by a factor of four and its volume by a factor of eight.

3.2.1 Geometric Scaling and Flight Muscle Fractions in Birds

Geometric scaling has been used quite successfully in many spheres of biology, for example, for comparing the effects of size and age in animals of the same species, and for comparing qualities and attributes in different species of animals. As an instance of the latter, consider Figure 3.2 wherein are plotted the total weight of the ßight muscles, W_{fm}, of quite a few birds against their respective body weights, W_b. How many birds are Òquite a fewÓ? The Þgure caption states that the underlying study actually included 29 birds, but the Þgure shows data only within the range $10 \leq$ bird number ≤ 23. For the 14 birds shown in Figure 3.2 there seems to be a fairly nice straight line Þt for the data presented. While Þtted by eye, that straight line can be determined to be:

$$W_{fm} \cong 0.18 W_b. \tag{3.2}$$

Equation (3.2) suggests that ßight muscle makes up about 18% of a birdÕs body weight and that ßight muscle weight *scales linearly* withÑor is

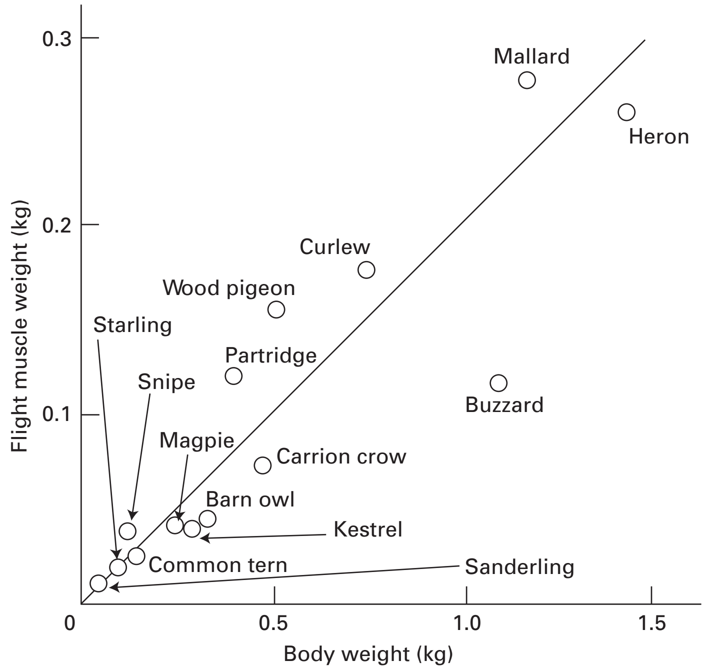

Figure 3.2 A simple linear fit on a plot of the total weight of the flight muscles against body weight for 14 of the 29 birds studied, including starlings, barn owls, kestrels, common terns, mallards, and herons (after Figure 1–2 of Alexander, 1971).

proportional toÑ body weight, a result that seems reasonable enough from our everyday observations of the birds around us.

3.2.2 Linearity and Geometric Scaling

These straightforward geometric scaling arguments can also be used to demonstrate some ideas about linearity in the context of *geometrically similar* objects, that is, objects whose basic geometry is essentially the same. In Figure 3.3 we show two pairs of drinking glasses: One pair are right circular cylinders of radius r, the second pair are right circular inverted cones having a common semi-vertex angle α. If the Þrst pair are Þlled to heights h_1 and h_2 respectively, the total ßuid volume in the two glasses is

$$V_{cy} = \pi r^2 h_1 + \pi r^2 h_2 = \pi r^2 (h_1 + h_2). \tag{3.3}$$

Equation (3.3) demonstrates that the volume is *linearly proportional* to the height of the ßuid in the two cylindrical glasses. Further, since the total

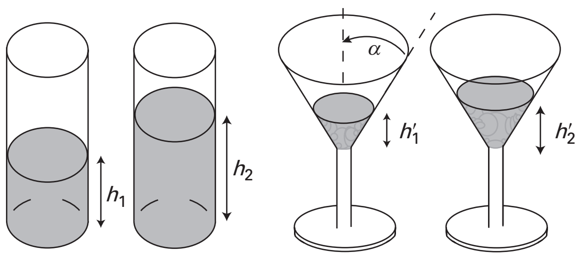

Figure 3.3 Two pairs of drinking glasses: One pair are cylinders of radius r, the second pair are inverted cones (sometimes referred to as martini glasses) having a common semi-vertex angle α.

volume can be obtained by adding or *superposing* the two heights, the volume V_{cy} is a *linear function* of the height h. (Recall the discussion in Section 1.3.4.) Note, however, that the volume is *not* a linear function of the radius, r.

For the two conical glasses, we see that their radii vary with height. In fact, the volume, V_{co}, of a cone with semi-vertex angle, α, Þlled to height, h, is

$$V_{co} = \frac{\pi}{3}\frac{h^3}{\tan^2\alpha}. \tag{3.4}$$

Hence, the total volume of ßuid in the two conical glasses of Figure 3.3 is

$$V_{co} = \frac{\pi}{3}\frac{h_1'^3}{\tan^2\alpha} + \frac{\pi}{3}\frac{h_2'^3}{\tan^2\alpha} \neq \frac{\pi}{3}\frac{(h_1' + h_2')^3}{\tan^2\alpha}. \tag{3.5}$$

The relationship between volume and height is nonlinear for the conical glasses, so we cannot calculate the total volume just by superposing the two ßuid heights, h_1' and h_2'.

3.2.3 "Log-log" Plots of Geometric Scaling Data

We now choose to ask a question: What happened to the other 15 birds in the small scaling study of Section 3.2.1? (Among those discriminated against in Figure 3.2 are hummingbirds, wrens, robins, skylarks, vultures, and albatrosses.) These birds were not included because the bird weights studied spanned a fairly large range, which made it hard to include the heavier birds (e.g., vultures and albatrosses) in the plot of Figure 3.2 without completely squashing the data for the very small birds (e.g., hummingbirds and goldcrests). This suggests a problem in organizing and presenting data, in itself an interesting aspect of scaling.

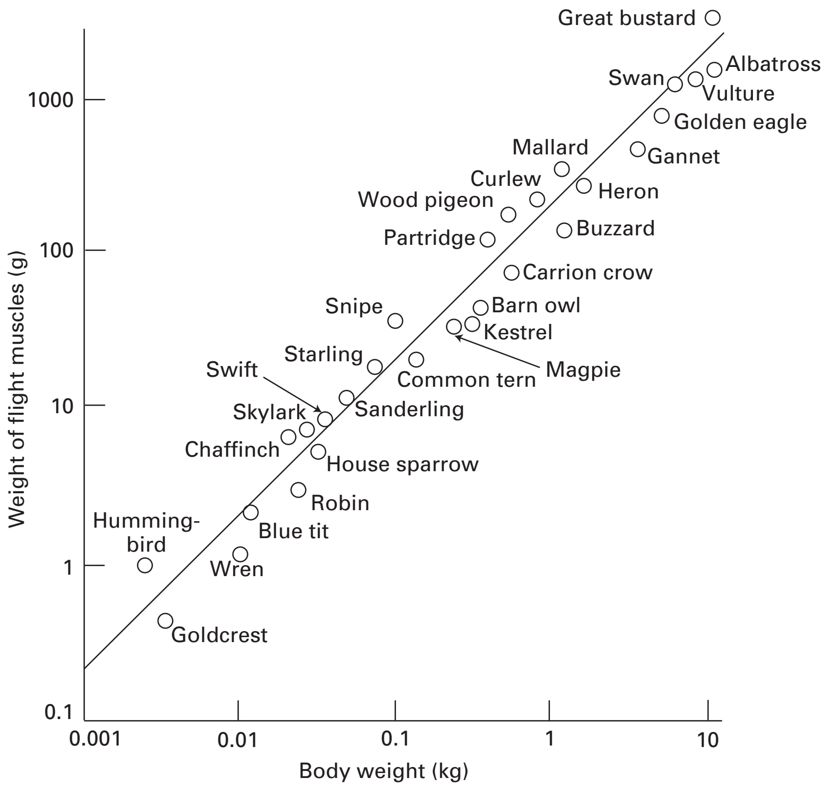

Figure 3.4 A "log-log" plot of the total weight of the flight muscles against body weight for 29 birds, including hummingbirds, wrens, terns, mallards, eagles, and albatrosses. Compare this with the linear plot of the data of Figure 3.2 (after Figure 1–4 of Alexander, 1971).

There is a straightforward way to include the heretofore left-out data: Construct *log-log* plots in which the *logarithms of the data* (normally to base 10) are graphed, as shown in Figure 3.4. In fact, the complete data set was plotted, essentially doubling the number of included data points, and a statistical regression analysis was applied to determine that the straight line shown in Figure 3.4 is given by:

$$W_{fm} = 0.18 W_b^{0.96}. \tag{3.6}$$

We could observe that eq. (3.6) is not exactly linear because, after all, $0.96 \neq 1$. However, it is clear that eqs. (3.2) and (3.6) are sufÞciently close that it is still quite reasonable to conclude that ßight muscle weight scales linearly with body weight.

However, this second look at the ßying muscle weight of birds raises two interesting scaling issues of scaling: First, how do we handle nonlinearities?

Second, how do we handle large ranges of data? In fact, we have just seen that these two questions are not unrelated because we found the *almost linear*, small nonlinearity in eq. (3.6), as a result of looking at an extended range of data.

We also have already provided an answer to the second question, namely, introducing log-log plots to extend our graphical range. Of course, with modern computational capabilities, we could skip the Òold fashionedÓ method of laboriously plotting data and simply enter tables of data points and let the computer spit out an equation or a curve. But something is gained by thinking through these issues without a computer.

Consider the data that emerged from a study of medieval churches and cathedrals in England. Large churches and cathedrals of that area (see Figures 3.5) were generally laid out in a cruciform pattern (viz., Figure 3.6) so that the *nave* was the principal longitudinal area, extending from a front entrance to a *chancel* or altar area at the back, and the *transept* was set out as a section perpendicular to the nave, quite close to the chancel. Was the cruciform shape ecclesiastically motivated, that is, was it inspired by religious feeling? In fact, research suggests that scaling dictated the

(a) (b)

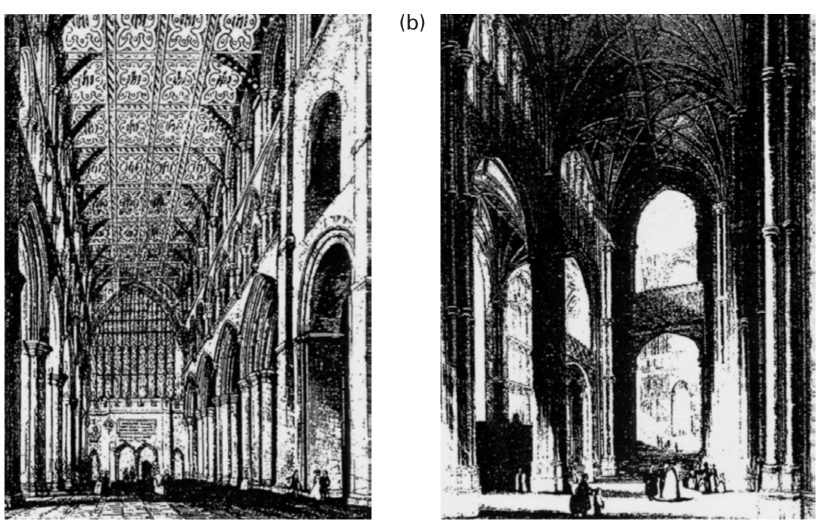

Figure 3.5 Interior views of two church *naves*: (a) The oldest Romanesque cathedral in England, St. Albans, has a nave with a relatively low height-to-length ratio; (b) The late Gothic style, also called the *perpendicular style*, is exemplified by the Canterbury Cathedral, whose nave has a relatively high height-to-length ratio (used by permission of the late Professor S. J. Gould of Harvard University).

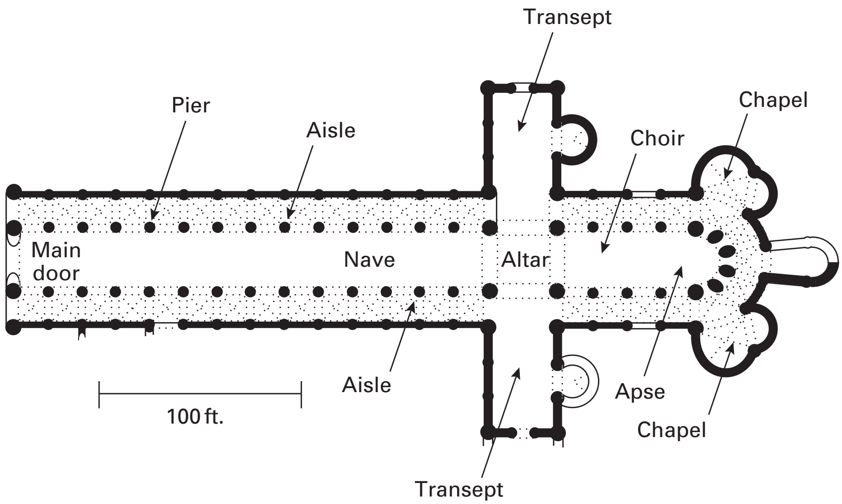

Figure 3.6 The plan of the Norwich Cathedral showing its cruciform shape, including the longitudinal nave, extending from the front door (left) to the rear apse (right), and the perpendicular transept (after Gould, 1975).

cruciform shape, and that the scaling was inspired by the need for both good lighting and sound structures.

We start by taking the length of a church as the Þrst-order indicator of its size. Thus, the longer its length, the larger the church. Then we examine the data displayed in Figure 3.7, which is a log-log plot of nave height against church length for a variety of medieval cathedrals and churches in England and on the European continent. We see from that data that as church length (and size) increase, the heights of their naves increases in absolute terms but falls off in *relative* terms. That is, as churches get longer (and larger), their naves get relatively smaller. Further, although we do not give the data to buttress this assertion, the bigger churches tend to have *narrower* naves. *Why don't the nave height and width increase with church size?* The answer has to do with the scaling of surface areas and enclosed volumes, that is, with geometric scaling.

The relevant scaling refers to the change in the area enclosed in a church as it is made longer (and larger). A longer church has a longer perimeter. In buildings of constant shape, the surface area of the enclosing wall increases linearly with the perimeter length, while the enclosed volume increases as (perimeter length)^2. Thus, problems emerge as it becomes more difÞcult for light and fresh air to penetrate into the churchÕs interior as its perimeter increases. (Remember that these marvelous structures were built long before the invention of the light bulb and air conditioning!) However,

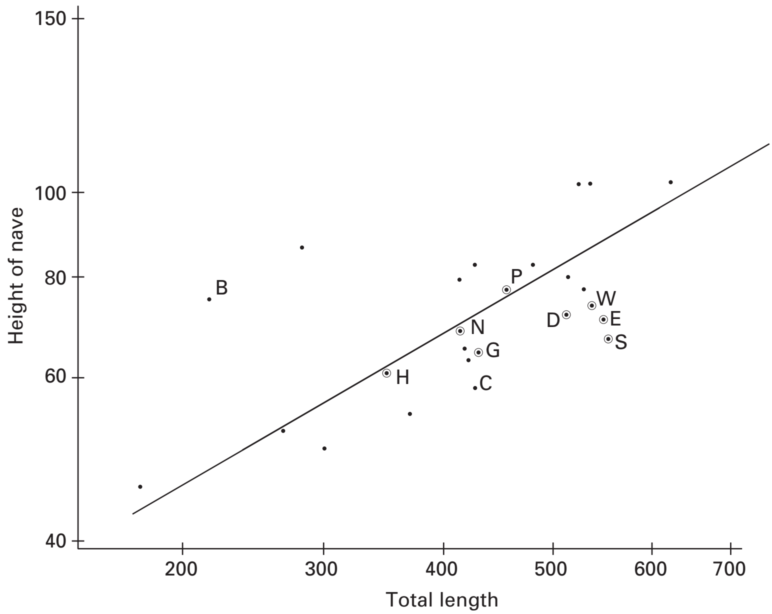

Figure 3.7 A plot of log (church nave height) against log (church length), with circled dots indicating Romanesque churches and letters standing for English churches as follows: B, Earls Barton; C, Chichester; D, Durham; E, Ely; G, Gloucester; H, Hereford; N, Norwich; P, Peterborough; S, St. Albans; and W, Winchester (used by permission of the late Professor S. J. Gould of Harvard University).

the severity of the lighting and ventilation problems can be reduced by introducing the transept because it enables a relatively constant nave width, thus taking away the Òconstant shapeÓconstraint. If the width is kept constant, then the enclosed area increases linearly with perimeter length, as does the churchÕs length (and size). And, of course, such a church will then appear to be relatively narrow!

Increasing a naveÕs width along with its length is another way to increase the enclosed area, but this approach also exacerbates interior lighting and ventilation problems. And it creates still another problem, namely that of building a roof with a larger surface area to cover the enlarged, enclosed area. Since roofs of cathedrals and churches were built to sit atop stone *vaults* and arches, roof spans became critical because it was very hard to build wide stone arches and vaults. The difÞculty of building wide arches also interacts with the height of the nave for it is the nave walls that support the outward thrust developed in the roof vaults, even when the nave walls are supported by ßying buttresses (see Figure 3.8). Thus, higher nave walls

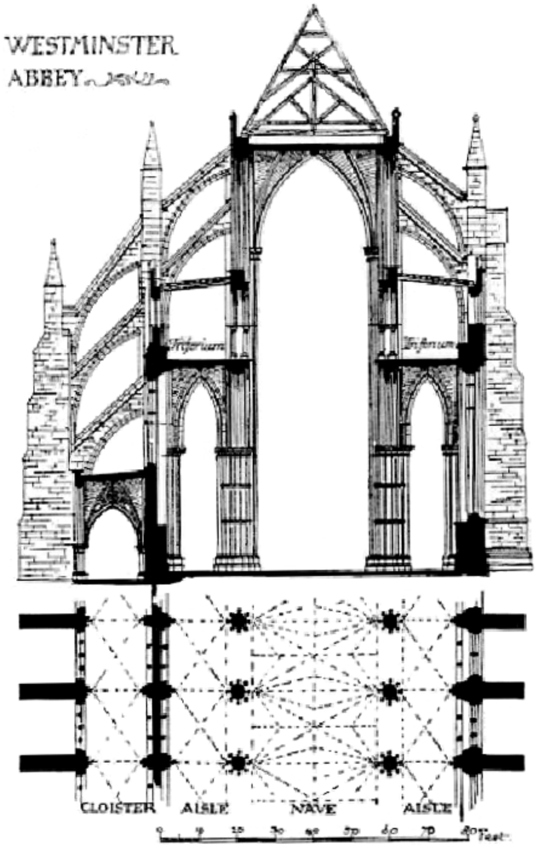

Figure 3.8 A cross-section of London's Westminster Abbey looking down the axis of the *nave*, showing the roof *vault* and the *flying buttresses* and their piers that support both roof vaults and nave walls. On the left (south) side there are *two* sets of flying buttresses, and the main buttressing piers are located beyond the *cloister* that abuts the church along that side (after Heyman, 1995).

had to be thicker to support both their own weight and the weight of the roofs supported by the vaults, which were in turn supported at the more-ßexible tops of the walls. Therefore, in sum, the width and height of cathedral naves had to be scaled back as overall church length (or size) was increased lest problems of lighting, ventilation, *and* structural safety become insoluble.

Problem 3.1. On what basis did the Lilliputians conclude that Gulliver needed 1728 times as much food as they did?

Problem 3.2. How would the LilliputiansÕconclusion change if they had thought about the exchange of energy between a person (of any size) and the surrounding environment?

Problem 3.3. How do the surface area and volume of a sphere scale? Why? (*Hint*: Analyze spheres of radii 1 and R.)

Problem 3.4. Explain what would happen to an angle between two lines inscribed on a balloon as it was inßated to a radius R from a radius of 1.

Problem 3.5. ConÞrm that eq. (3.2) does adequately portray the straight line drawn in Figure 3.2.

Problem 3.6. Show how the equation $y = mx^b$ becomes a linear equation in a log-log plot.

Problem 3.7. Write eq. (3.6) in a form suitable for a log-log plot.

3.3 Size and Function–I: Birds and Flight

We now examine another set of empirical data, taken from a study of the aerodynamics of birds in ßight and displayed in Figure 3.9. It appears from this plot that a straight line can be penciled in to Þt the data, and it also seems that there is no data for bird weights greater than 35Ð40 pounds. We are thus prompted to ask two questions: Can the general form of this data be explained by dimensional analysis, along the lines of our discussions of Chapter 2? And, is there an upper limit to the weight of a ßying[1] bird? The answers to both questions are afÞrmative.

The answer to the Þrst question can be found by looking at the Þt of the straight line in the log-log plot of Figure 3.9. A close examination of the Þtted line shows that its slope is 1:3, which suggests that

$$\text{Weight} \propto (\text{Wing loading})^3. \tag{3.7}$$

But does eq. (3.7) make dimensional sense?

For birds that soar (e.g., gulls and buzzards), we argue that the lift forces needed to sustain them in the air should be proportional to the wing areas, or in dimensional terms, proportional to $(\text{length})^2$. The *wing loading* is

[1] Remember that several creatures categorized as birds have never taken wing, including penguins and ostriches, so we really do need the adjective Òßying.Ó

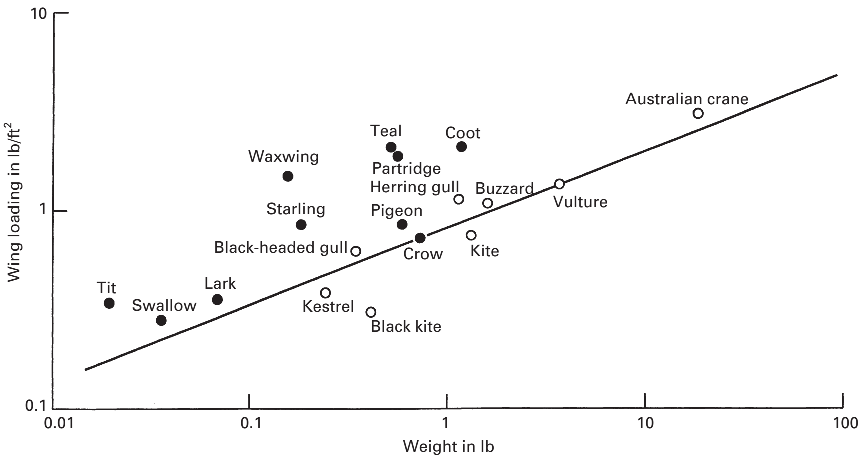

Figure 3.9 A set of empirical data, taken from a study of the aerodynamics of birds in flight (von Karman, 1954). It appears from this plot that a straight line can be penciled in to fit the data, and it also seems that there is no data for bird weights greater than 35–40 lb.

the load a bird has to carry, which is just its weight, which is proportional to its volume. Thus, in dimensional terms, the wing loading is proportional to (length)^3. Then the wing loading per unit of wing area would be proportional to (length)^3/(length)^2, or to (length). Since the weight is proportional to (length)^3 and the wing loading to (length), eq. (3.7) is dimensionally consistent.

The second question, about the existence of an upper bound to ßying weight, is harder to answer. We will answer it, but in the somewhat restricted domain of *hovering flight* because the aerodynamic arguments are simpler. We will formulate the problem by examining the dimensions of both the power *needed* to sustain hovering and the power *available* to sustain hovering.

3.3.1 The Power *Needed* to Hovering

A bird ßaps its wings when it is hovering. In so doing, the bird generates the needed hovering power by moving a mass of airÑ and so transferring momentumÑ downward. NewtonÕs second law says that the time rate of change of the momentum of that jet of air must be equal to the total lift force on the wings, which is, in turn, equal to the birdÕs weight. The mass of air moving through the jet can be estimated in terms of the air density, ρ, the wing area, A, and the jet speed, v, as

$$\text{mass/time} = \rho A v. \tag{3.8}$$

The time rate of change of momentum is then just the product $v \times$ (mass/time), which is, again, equal to the bird weight, W:

$$W = v \times \text{mass/time} = \rho A v^2. \tag{3.9}$$

In view of the dimensional dependencies of the birdÕs weight and wing area, it follows from eq. (3.9) that the velocity of the air mass for hovering must be such that

$$v \propto \mathsf{L}^{1/2}. \tag{3.10}$$

The power needed to sustain the hovering jet is equal to the time rate of change of the kinetic energy of the mass of air in the jet. Thus,

$$\text{power needed} \propto \tfrac{1}{2}\rho A v \times v^2. \tag{3.11}$$

In view of eqs. (3.10) and (3.11) taken together, the scaling of the power needed for a bird to hover scales with length according to:

$$\text{power needed} \propto \mathsf{L}^{7/2}. \tag{3.12}$$

Equation (3.11) is valid for forward ßight as well as hovering. It can be conÞrmed by more complete, more complex aerodynamic arguments.

3.3.2 The Power *Available* for Hovering

There are three ways we can estimate the power available to a bird to enable it to hover. We can estimate its heat loss during hovering, the rate at which its heart supplies oxygen, and the maximum stresses in its bones and muscles.

The *heat loss* estimate is simple, if not altogether compelling. Muscles turn chemical energy into mechanical energy at a 25% efÞciency rate. The excess energy is dissipated as heat loss through the birdÕs surface area. The heat transfer thus decreases at a rate proportional to (length)^2. Hence, in order to prevent the bird from overheating, the available power must also be proportional to L^2.

The *oxygen supply* estimate reduces to the consideration of the time rate of change of the volume of blood delivered by the heart. This volumetric rate is proportional to the cross-sectional area of the birdÕs blood vessels. Thus, we again Þnd that the available power is proportional to L^2 because it is proportional to the oxygen ßow, which is in turn proportional to the rate of blood delivery.

The *maximum stress* estimate begins with the assessment of the work done by a contracting muscle. By the principle of conservation of energy that work must equal the resulting change in the kinetic energy of the limb moved by the muscleÕs contraction. Thus,

$$\text{muscle force} \times \text{contraction} \propto \text{limb mass} \times v^2, \qquad (3.13)$$

where v is now the speed of the moving limb. Since the force in the muscle is limited by the maximum tensile strengths of the birdÕs muscles and tendons, it must be proportional to L^2 as representative of the cross-sectional area of those muscles and tendons.

Now the muscle contraction is proportional to L, and the limb mass to L^3, so that eq. (3.13) tells us that the speed of the hovering bird is independent of L or size. If this is true, the time it takes for a muscle to contract would be found from the ratio L/v, or simply the length L. Then the power exerted by the muscle is

$$\text{power} \propto \frac{\text{muscle force} \times \text{contraction}}{\text{time}} \propto \frac{L^2 \times L}{L}, \qquad (3.14)$$

so once again we Þnd that the available power is proportional to L^2.

3.3.3 So There Is a Hovering Limit

We have seen in Section 3.3.1 that the power needed for ßight is proportional to $L^{7/2}$, while in Section 3.3.2 we showed that the power available to the bird to sustain ßight is proportional to L^2. Since the power needed to hover increases so much faster with the bird size, it is clear that a limit to hovering size must indeed exist.

Problem 3.8. ConÞrm the dimensional relationship of eq. (3.10).

Problem 3.9. Use dimensional analysis to conÞrm that power is equal to the time rate of change of kinetic energy.

Problem 3.10. ConÞrm the dimensional relationship of eq. (3.12).

Problem 3.11. ConÞrm that eq. (3.13) does show that v is independent of L.

3.4 Size and Function–II: Hearing and Speech

Human hearing and speech are areas of human physiology where scaling has interesting and important effects. Size, shape, and function are clearly

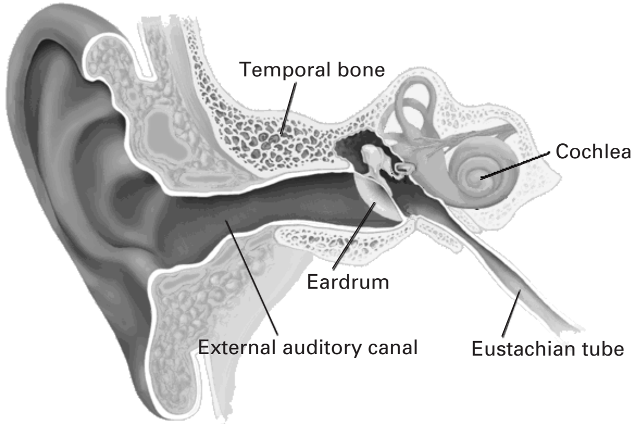

Figure 3.10 A cross-section of the human ear, including the eardrum, which is the mechanism with which we hear (from Encyclopedia Britannica Online, www.brittanica.com, 1997).

intertwined in the ear and eardrum (Figure 3.10), and in the vocal cords and *larynx*, that is, the Òvoice boxÓ that contains the vocal cords (Figure 3.11). We are inclined to wonder about scale effects in hearing because we know that humans hear sounds in the range of 20 to 20,000 Hz (or hertz or cycles per second), dogs hear sounds that have frequency components up to 50,000 Hz, and bats hear sounds as highs as 100,000 Hz. The unit hertz is named after the acoustician Gustav Ludwig Hertz (1887Ð1975). Since larger animals seem to have more limited frequency ranges, it is worth exploring whether size could play a role in these differences.

3.4.1 Hearing Depends on Size

The eardrum is just one part of a complex hearing apparatus (see Figure 3.10) that starts at the outer ear and goes through the cochlea to the auditory nerve that transmits signals to the brain for interpretation. When a sound is generated by a source, the result is that air (or another mediumÕs) particles immediately adjacent to the source are set into motion, creating the acoustic signals that are transmitted through the intervening air (or medium, or media) to the receiverÕs ear. The eardrum itself converts the mechanical vibration of the ÒincomingÓ air particles that form the acoustic signal into a mechanical vibration of three bonesÑ called the *hammer, anvil*, and *stirrup*Ñ that in turn carry the vibratory signal into the inner ear. Eventually, the inner ear converts these mechanical signals into electrical signals that are transmitted through the nervous system to the

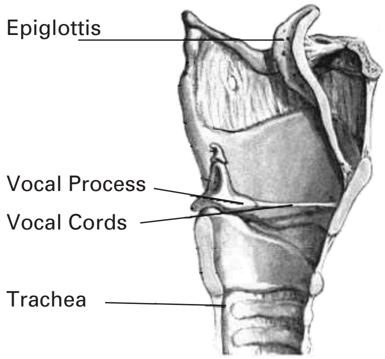

Figure 3.11 A cross-section of the human *larynx* or "voice box" showing the vocal cords that are the mechanism with which we speak (www.sfu.ca/~saunders/L33098/L5/L5Fset.html, 2002, by courtesy of R. Saunders, Simon Fraser University, Burnaby, British Columbia, Canada).

brain by way of the organs of Corti. As the Þrst pickup of the incoming mechanical signal, it is important that the eardrum remain quite stiff in order to pick up and accurately reproduce the higher frequencies of that signal.

In mechanical terms, the eardrum is a stretched membrane, much like a trampoline. Like every other mechanical device, the eardrum has *natural frequencies* at which it can vibrate freely (and indeÞnitely, if only there were no damping!). As we will see in Chapter 8, an elastic system responds just like a linear spring when it is forced or excited at frequencies below the lowest natural frequency, sometimes called the *fundamental frequency*. Thus, if we want the eardrum to be stiff, we want its fundamental frequency to be very high. It turns out that the fundamental frequency of a stretched circular membrane of radius, r, and thickness, h, is given by

$$f_{membrane} = \frac{2.40}{2\pi r}\sqrt{\frac{F}{\rho h}}, \tag{3.15}$$

where F is the tensile (stretching) force per unit length of the membrane circumference and ρ is the mass density of the material of which the membrane is made. It is easily veriÞed that the dimensions of the membraneÕs fundamental frequency are 1/(time) or 1/T, which is quite appropriate for frequency. However, it is also interesting to note in eq. (3.15) that the fundamental frequency varies inversely with the radius, r, of the membrane (or eardrum). So, for similar values of the tensile force, F, and the mass density, ρ, we would expect the range of hearing to extend into higher frequencies for smaller animals, and this is just what we have seen in the hearing ranges of humans, dogs, and bats.

3.4.2 Speech Depends on Size

A similar situation exists with regard to human vocal cords and voice boxes. We know from everyday experience that men generally have deeper, lower-pitched voices than do women, and we are also accustomed to the facts that birds chirp and bears growl. So we are inclined to imagine that the sound of speech would scale with size.

The mechanism that creates speech is the forced vibration of the vocal cords as air is expelled from the lungs and pushed past (and through) the cords in the *larynx* or voice box (viz., Figure 3.11). In order to develop and produce volume at low frequencies, the vocal cords must be able to vibrate at low frequencies, and the voice box must be able to amplify the low-frequency signals produced by the vocal cords.

The vibration characteristics of vocal cords can be modeled just as we would model the vibration of violin or piano strings, whose fundamental frequency is given by

$$f_{string} = \frac{1}{2l}\sqrt{\frac{F}{\rho A}}, \tag{3.16}$$

where l is the stringÕs length, A its cross-sectional area, F is the tensile force applied to the string, and ρ its mass density. Note the very strong resemblance between eqs. (3.15) and (3.16). Further, we see that this frequency (3.16) scales inversely with both the string length and its mass density. Thus, a larger animal with longer and more dense vocal cords will make sounds that have components at lower frequencies.

We can also look at the fundamental frequency of an *acoustic cavity* as a model for the larynx. Such a cavity, which we examine in more detail in Chapter 8, is also called an *acoustic resonator* and it has a fundamental frequency given by [see eq. (8.47)]

$$f_{cavity} = \frac{c_0}{2\pi}\sqrt{\frac{A}{lV_0}}, \tag{3.17}$$

where A and l are, respectively, the area and the length of the ÒneckÓ leading into an acoustical cavity of volume V_0 that is Þlled with a gas in which sound waves travel at a speed c_0 (see Figure 8.7). Clearly, the fundamental frequency of the cavity scales inversely with the cavityÕs volume. So once again we Þnd that larger humans and animals have deeper voices than do their smaller counterparts.

Problem 3.12.	What is the hearing range of an elephant? A whale? How do these ranges compare with those of humans?
Problem 3.13.	ConÞrm that the dimensions of eq. (3.15) are 1/T.
Problem 3.14.	ConÞrm that the dimensions of eq. (3.16) are 1/T.
Problem 3.15.	ConÞrm that the dimensions of eq. (3.17) are 1/T.

3.5 Size and Limits: Scale in Equations

In Section 3.3, while discussing size and function, we found that there is an upper limit to the weights of hovering birds. This limit is due to the fact that birds could not supply enough power to sustain hovering ßight as they grew bigger and heavier. Thus, the birdsÕ ability to hover was limited by the power available to them. Limits occur quite often in mathematical modeling, and they may control the size and shape of an object, the number and kind of variables in an equation, the range of validity of an equation, or even the application of particular physical modelsÑor Òlaws,Ó as they are often called.

Modern electronic components and computers provide ample evidence of how limits in different domains have changed the appearance, performance, and utility of a wide variety of devices. The bulky radios that were made during the 1940s, or the earliest television sets, were very large because their electronics were all done in old-fashioned circuits using vacuum tubes. These tubes were large and threw off an enormous amount of heat energy. The wiring in these circuits looked very much like that in standard electrical wiring of a house or ofÞce building. Now, of course, we carry television sets, personal digital assistants (PDAs), and wireless telephones on our wrists. These new technologies have emerged because we have learned to dramatically change the limits on fabricated electrical circuits and devices, and on the design and manufacturing of small mechanical objects. And this is true beyond electronics. The scale at which surgery is done on people has changed because of our ability to ÒseeÓ inside the human body with greater resolutionÑwith increasingly sophisticated scans and imagers, as well as with Þber-optic television camerasÑand to design visual, electronic, and mechanical devices that can operate inside a human eye, and in arteries and veins. In the emerging Þeld of *nanotechnology* we are learning to engineer things at the molecular level. Thus, our mathematical models will change, as will the resulting devices and Òmachines.Ó

3.5.1 When a Model Is No Longer Applicable

As we hinted in Section 3.1, one interesting example of the interaction of scale and limits is Newtonian mechanics. We are accustomed to taking the masses or weights of objects as constants in our everyday lives and in our normal engineering applications of mechanics. We do not expect a box of candy to weigh any more whether we are standing still, riding in a car at 120 km/hr (75mph), or ßying across the country at 965 km/hr (600 mph). Yet, as we noted in Section 3.1, according to the general theory of relativity, the mass of a particle moving at speed, v, is given as a (dimensionless) fraction of the rest mass, m_0, by

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-(v/c)^2}}, \tag{3.18}$$

where c is the speed of light (3×10^8 m/s $=$ 186,000 mi/sec). For the box of candy ßying across the country at 965 km/hr $=$ 268 m/s, the factor in the denominator of the relativistic mass formula (3.18) becomes

$$\sqrt{1-\left(\frac{v}{c}\right)^2} = \sqrt{1-7.98 \times 10^{-13}} \cong 1 - 3.99 \times 10^{-13} \cong 1. \tag{3.19}$$

Clearly, for our practical day-to-day existence, we can neglect such relativistic effects. However, it remains the case that Newtonian mechanics is a good model only on a scale where all speeds are very much smaller than the speed of light. If the ratio v/c becomes sufÞciently large, the mass can no longer be taken as the constant rest mass, m_0, and Newtonian mechanics must be replaced by relativistic mechanics.

3.5.2 Scaling in Equations

In certain situations, scaling may shift limits or perhaps points on an objectÕs boundary where *boundary conditions* are applied. For example, suppose we want to approximate the hyperbolic sine function,

$$\sinh x = \frac{1}{2}(e^x - e^{-x}). \tag{3.20}$$

We know that for large values of x, the term e^x will be much larger than the term e^{-x}. The approximation problem is one of deÞning an appropriate criterion for discarding the smaller term, e^{-x}. For dimensionless values of x greater than 3, the second term on the right-hand side of eq. (3.20), e^{-x}, does become very small (less than 4.98×10^{-2}) compared to e^x for $x = 3$, which is 20.09. Hence, we could generally take $\sinh x \cong (½)e^x$. All we have to do is decide on a value of x for which we are willing to accept the approximation $e^{2x} - 1 \cong e^{2x}$.

We can also approach this problem by introducing a *scale factor*, λ, after which we can look for values of x for which we can make the approximation

$$\sinh(x/\lambda) \cong \frac{1}{2} e^{x/\lambda}. \tag{3.21}$$

Putting a scale factor, λ, in the approximation of eq. (3.21) obviously means that it will affect the value of x for which that approximation is acceptable. Now the comparison is one in which we want

$$e^{2x/\lambda} - 1 \cong e^{2x/\lambda}. \tag{3.22}$$

For $\lambda = 1$, the approximation is good for $x \geq 3$, while for $\lambda = 5$ the approximation works for $x \geq 15$. Thus, by introducing the scale factor λ we can make the approximation valid for different values of x because we are now saying that $e^{-x/\lambda}$ is sufÞciently small for $x/\lambda \geq 3$. Changing λ has in effect changed a boundary condition because it has changed the expression of the boundary beyond which the approximation is acceptable to $x \geq 3\lambda$.

Recall that functions such as the exponentials of eqs. (3.21) and (3.22), as well as sinuosoids and logarithms, are *transcendental functions.* Transcendental functions can always be represented as power series, as we will detail in Section 4.1.2. For example, the power series for the exponential function is:

$$e^{x/\lambda} = 1 + \frac{x}{\lambda} + \frac{1}{2!}\left(\frac{x}{\lambda}\right)^2 + \frac{1}{3!}\left(\frac{x}{\lambda}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{x}{\lambda}\right)^n + \cdots \tag{3.23}$$

It is clear that the argument of the exponential must be dimensionless because without this property eq. (3.23) would not be a rational equation. Further, we could not calculate numerical values for the exponential function, or any other transcendental function, if its argument was not dimensionless. The presence of a scale factor in eq. (3.22) makes the exponential $\tilde{\text{Õ}}$ argument dimensionless, and so numerical calculations can be performed.

In addition, the scale factor, λ, often represents a characteristic aspect of the problem being modeled, so that a ratio such as x/λ becomes a useful measure of whether something is truly large or small. For example, the hyperbolic sinusoid in eq. (3.20) might describe the deßection or downward displacement of a catenary cable as a function of its length.

The variable x could be a coordinate measured along the projected cable length and λ could represent its total projected length, which could be regarded as the cableÕs *characteristic length.*

3.5.3 Characteristic Times

We often see rate effects in Þrst-order differential equations (a brief review of which can be found in Section 5.2.2). For example, it will be shown that a charged capacitor draining through a resistor produces a voltage drop $V(t)$ at a rate proportional to the actual value of the voltage at any given instant. The mathematical model would be:

$$\frac{dV(t)}{dt} = -\lambda V(t). \tag{3.24}$$

We can rewrite this equation in the equivalent form

$$\frac{dV(t)}{V(t)} = -\lambda dt. \tag{3.25}$$

Now, in order for this rate equation to be a rational equation, the net dimensions of each side of the equation must be the same. For eq. (3.25) that means each side must be dimensionless. The left-hand side is clearly dimensionless because it is the ratio of a voltage change to the voltage itself. The right-hand will be dimensionless only if the scale factor, λ, has physical dimensions such that $[\lambda] = 1/\mathsf{T}$. We will soon see this below and then will reconÞrm it when we solve the differential equation (3.24) in Chapter 5.

We can use the dimensionless product λt to derive a measure of the time that it takes to discharge the capacitor being modeled. For example, we could deÞne a *decay time*, often called a *characteristic time*, as the time it takes for the voltage to decrease to a speciÞed fraction of its initial value. Suppose we choose that speciÞed value to be 1/10. The characteristic or decay time of the charged capacitor would then be

$$V(t_{decay}) \equiv \frac{V_0}{10}. \tag{3.26}$$

How would we calculate t_{decay}? As we will show in Chapter 6, it is easily conÞrmed that the solution to the differential equations (3.24) and (3.25) is

$$V(t) = V_0 e^{-\lambda t}, \tag{3.27}$$

which in view of eq. (3.26) means that

$$\lambda \cong \frac{2.303}{t_{decay}}. \tag{3.28}$$

Equation (3.28) simply says that the scale factor λ for the discharging capacitor is inversely proportional to the characteristic (decay) time, and that the voltage in the capacitor can then be written as

$$V(t) \cong V_0 e^{-2.303(t/t_{decay})}. \tag{3.29}$$

Problem 3.16.	Under what conditions is eq. (3.24) dimensionally consistent?
Problem 3.17.	ConÞrm that the voltage of eq. (3.27) does satisfy eq. (3.24).
Problem 3.18.	ConÞrm that eq. (3.28) is correct.

3.6 Consequences of Choosing a Scale

Since all actions have consequences, it should come as no surprise that the acquisition of experimental data, its interpretation, and its perceived meaning(s) generally can be very much affected by the choice of scales for presenting and organizing data.

3.6.1 Scaling and Data Acquisition

Scales affect the ways in which data is taken during experiments. Carefully chosen scales can reduce errors, save time and money, and they can highlight important details.

Consider, for example, the simple apparatus shown in Figure 3.12, which can be used to determine the *rotational inertia*, I_{rot}, (the second moment of inertia of the mass around an axis through its centroid or center) of the wheel shown as it turns or spins around an axis through its center. This experiment uses a falling weight connected to the wheel by a string to produce a torque that, in turn, causes the wheel to rotate. That torque, τ, is related to the rotational inertia I_{rot} by

$$I_{rot} = \frac{\tau}{\alpha}, \tag{3.30}$$

where α is the angular acceleration of the wheel, measured in units of radians per second squared (rad/sec^2). As we describe the inßuence of scale on experimental observation, we will focus on the angular acceleration as the important parameter through which we can determine I_{rot}. We conduct the experiment itself by letting the falling weight cause the wheel to spin, during which we measure or read the speed, v, of any point on the wheel both as the experiment begins at some time $t = t_0$, and at a later time ($t = t_f$) that denotes the end of the experiment. The angular acceleration is then calculated in terms of the wheel©radius, R, and the measured speeds and measurement times as:

$$\alpha_{exp} = \frac{(v_f - v_0)}{R(t_f - t_0)}, \tag{3.31}$$

where the speeds v_0 and v_f are measured at the times t_0 and t_f, respectively.

Clearly, the time scale for this experiment is the time interval $t_f - t_0$. It will control the amount of error between the experimentally determined value of I_{rot} and its actual (or theoretically calculated) value.

We know that the wheel is set into motion by releasing or dropping the falling weight, because that action pulls the string taut and causes the wheel to start spinning. As the weight falls, the wheel rotates at an increasingly faster rate. Since the wheel is at rest when we initiate each experimental run, we can safely take $t_0 = v_0 = 0$. Then the values of α_{exp} determined experimentally are found from eq. (3.31) as

$$\alpha_{exp} = \frac{v_f}{Rt_f}. \tag{3.32}$$

Now, while we have argued above that the apparatus shown in Figure 3.12 produces a constant acceleration, that is not exactly true. Since we are starting from the state $t_0 = v_0 = 0$, static friction must be overcome as the wheel starts from rest at the beginning of each run of the experiment. After a short while, the wheel motion does, in fact, settle into spinning with a fairly constant acceleration. But what exactly is a Òshort whileÓ How do we know the correct value of t_f at which we can terminate each experimental run? Is 2 seconds enough time? Or do we need 4 seconds, or a still longer time?

In Table 3.1 we show some data obtained in one run of this experiment. Note that the number of revolutions or spins of the wheel goes up rapidly as time elapses, as does the speed of rotation. Further, and most importantly, if we calculate the angular acceleration as it varies with time (or with the estimated number of revolutions, a number that we can also count), we see that α_{exp} appears to approach a constant value (which means the torque also approaches a constant value). Why is this so? It is so because when we allow the experiment to run for a longer time (or through more turns of the wheel), we are changing the time scale over which the drag due to static friction has an inßuence. In a very short experiment, the time taken to overcome static friction takes up a much larger percentage of the time scale

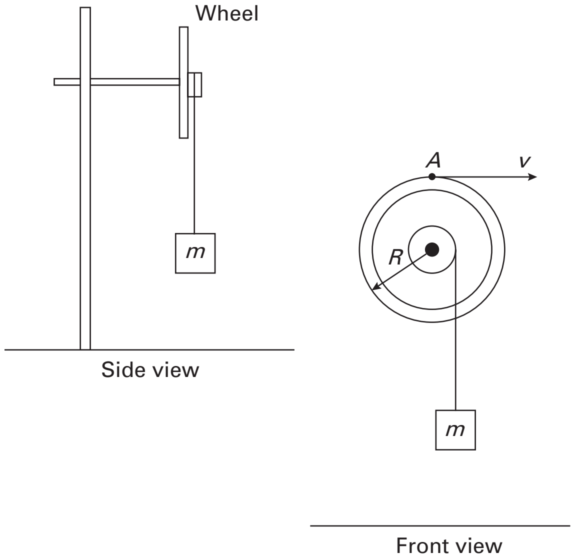

Figure 3.12 A simple piece of apparatus that can be used to measure the rotational inertia of a wheel of radius, R, as it spins around an axis through its center.

of the experiment, and so it has a disproportionate inßuence. In a longer experiment, conditions approach a steady state in which the predominant effect is the torque applied by the falling weight, so the static friction occupies an increasingly smallÑ and negligibleÑ part of the experimentÕs run time.

Table 3.1 The data taken in the experimental determination of the rotational inertia of the wheel (as shown in the apparatus of Figure 3.12), along with an estimate of the actual number of revolutions that had occurred when v_f was measured.

t_f (s)	Estimated number of revolutions	Measured v_f (m/s)	Calculated $\alpha_{exp} = v_f/Rt_f$ (rad/s^2)
2	1/5	0.55	0.55
6	2.4	2.47	0.82
10	7	4.48	0.90
20	30	9.50	0.95
100	790	49.68	0.99

As another illustration of how scaling affects data acquisition, consider the diagnosis of a malfunctioning electronic device such as an audio ampliÞer. Such ampliÞers are designed to reproduce their electrical input signals without any distortion. The outputs are distorted when the input signal has frequency components beyond the ampliÞerÕs range, or when the ampliÞerÕs power resources are exceeded. Distortion also occurs when an ampliÞer component fails, in which case we must diagnose the failure to identify the particular failed component(s).

A common approach to doing such diagnoses is to display (on an oscilloscope) the deviceÕs output to a known input signal. If the device is working properly, we expect to see a clear, smooth replication of the input. One standard test input is the square wave shown in Figure 3.13 (a). A nice

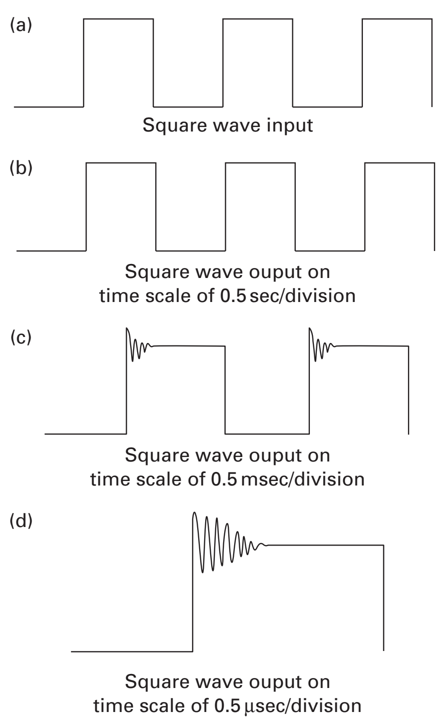

Figure 3.13 A square wave (a) is the input signal to a (hypothetical) malfunctioning electronic device. Traces of the output signals are shown at three different time scales (i.e., long, short, shorter):
(b) 0.5 second/division;
(c) 0.5 millisecond/division; and
(d) of 0.5 microsecond/division.

replication of that square wave is shown in Figure 3.13(b), and it seems just Þne until we notice that the horizontal time scale is set at a fairly high value, that is, 0.5 second/division. To ensure that we are not overlooking something that might not show up on this scale, we spread out the same signal on shorter time scales of 0.5 millisecond/division (Figure 3.13(c)) and 0.5 microsecond/division (Figure 3.13(d)), neither of which is the nice sinusoid we originally thought. This suggests that the device is malfunctioning. Had we not set the oscilloscope to shorter, more appropriate time scales, we might have come to an erroneous conclusion.

3.6.2 Scaling and the Design of Experiments

Scale also affects the ways in which experiments are designed, especially when the context is that of ensuring that models replicate the prototypes or ÒrealÓ artifacts that they are intended to stand for or model. This aspect of scaling is, as we will now show, intricately intertwined with the notions of dimensional analysis discussed in Chapter 2.

Scale models or reproductions of physical phenomena or devices are usedÑ as they have been for quite some timeÑ to do experiments and study behavior for which a comprehensive analytical model is not available. Often such studies are done because a laboratory experiment is more easily developed than is a full-scale experiment. For example, it is easier to study the vibration characteristics of a model of a proposed bridge design than it is to build the designed bridge and hope for the best, just as it is easier to test models of rockets in simulated spaceßight or models of buildings in simulated earthquakes or Þres. But such experimental models wonÕt be of much use unless some preliminary analysis is done and clear physical hypotheses are developed in advance. We will illustrate how is that done with one simple example.

Consider a simple beam, such as the one shown in Figure 3.14.

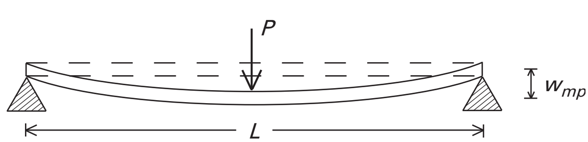

Figure 3.14 Prototype and model of a simple elastic beam of length, L, elastic modulus, E, and second moment of cross-section, I, as it deflects an amount, w_{mp}, at its center due to the load of magnitude, P, applied there.

We assume that it is known that the deßection, $w(L/2)$, of the midpoint of the beam when concentrated load P is applied at the same point is a function of the load and three other parameters, that is,

$$w(L/2) \equiv w_{mp} = f(P, EI, L), \tag{3.33}$$

where L is the beamÕs length, E is the elastic modulus of the beam material, and I is the second moment of its cross-sectional area. The product EI is commonly called the beamÕs *bending stiffness.* This example has two dimensionless groups (see Problem 3.19):

$$\Pi_1 = \frac{w_{mp}}{L}, \quad \Pi_2 = \frac{PL^2}{EI}. \tag{3.34}$$

Thus, it follows that

$$\frac{w_{mp}}{L} = f\left(\frac{PL^2}{EI}\right). \tag{3.35}$$

Suppose we want to determine the functional form of eq. (3.35) for a beam, which we will call the *prototype* beam, but that the beam is too big and heavy, and the load P too large, for us to do an experiment on the beam itself. We propose instead to test a *model* beam. But then we immediately face a question: How should the properties of the model beam relate to those of the prototype? The answer lies in the results obtained by applying the principles of dimensional analysis: The model properties and prototype properties must be such that the two dimensional groups have the same numerical values for both model and prototype. Stated in mathematical terms, with subscripts ÒmÓ for model and ÒpÓ for prototype,

$$(\Pi_1)_m = (\Pi_1)_p, \quad (\Pi_2)_m = (\Pi_2)_p. \tag{3.36}$$

Thus, to a certain extent we can scale the geometry, the material, or the load for our own convenience, but we cannot scale all of the independent variables independently. In order to preserve the property of *complete similarity* between model and prototype, we must preserve the equality between model and prototype of each dimensionless group needed to deÞne a particular problem.

Applying the general similarity rule of eq. (3.36) to the speciÞc case of the beam whose dimensionless groups are given in eq. (3.34), we Þnd that we can preserve complete similarity by requiring that

$$\left(\frac{w_{mp}}{L}\right)_m = \left(\frac{w_{mp}}{L}\right)_p, \quad \left(\frac{PL^2}{EI}\right)_m = \left(\frac{PL^2}{EI}\right)_p. \tag{3.37}$$

Having established in eq. (3.37) the overall conditions needed for complete similarity, we can now go into further detail to see both what we *must* do and what we *may* do in terms of *scaling factors* deÞned for each of the problemÕs variables, that is, for the factors

$$n_w = \frac{\left(w_{mp}\right)_p}{\left(w_{mp}\right)_m}, \; n_P = \frac{P_p}{P_m}, \; n_E = \frac{E_p}{E_m}, \; n_I = \frac{I_p}{I_m}, \; n_L = \frac{L_p}{L_m}. \tag{3.38}$$

Thus, we see that the scaling factors in eq. (3.38)Ñ which should not be confused with the graphical scale factors, λ, introduced in Section 3.5.2Ñ are simply ratios of the values of each of the variables in the prototypes to the values of the same variable in the model. Equation (3.38) shows that we have Þve such scaling factors for this problem, while eq. (3.37) shows that there are two overall similarity conditions that must be satisÞed. We can, in fact, write the similarity conditions (3.37) in terms of the scaling factors (3.38) by straightforward substitution:

$$\frac{n_w}{n_L} = 1, \quad \frac{n_P n_L^2}{n_E n_I} = 1. \tag{3.39}$$

So, if we choose a length scale (n_L) for this problem, we have also chosen a deßection scale (n_w) by the Þrst of eq. (3.39). However, this means that we may still freely choose two of the three remaining scaling factors (n_P, n_E, and n_I). If we chose the scaling factors of the elastic modulus (n_E) and of the moment of inertia (n_I) because we had appropriate materials or small beams lying around our laboratory, then the single remaining scaling factor n_P would be determined by the second of eq. (3.39):

$$n_P = \frac{n_E n_I}{n_L^2}. \tag{3.40}$$

Suppose we wanted to model the deßection of a steel beam by doing experiments on a small model made of balsa wood. Assume a typical laboratory scenario in which the length scale is twenty-to-one, that is, $n_L = 20$, the scaling factor of the moments of inertia is about $n_I = 1000$, and that the scaling factor of the moduli of elasticity is approximately $n_E = 50$. For a similar experiment, we would then expect that the resulting deßection would be one-twentieth of the actual deßection when we apply a load to the model that is equal to the anticipated actual load divided by 125.

Note that this introduction to the consequences of scaling in modeling is just that, a very short and very limited introduction. Clearly, not all experiments are so easily analyzed or scaled, and so there are many more issues to be explored in a comprehensive look at scaling in the design of experiments.

3.6.3 Scaling and Perceptions of Presented Data

The scales used to present modeling ÒresultsÓalso signiÞcantly inßuence how such data is perceived, no matter whether those models are analytical or experimental in nature. Indeed, individuals and institutions have been known to choose scales and portrayals to disguise or even deny the realities they purport to present. Thus, whether by accident or by intent, scales can be chosen to persuade. While this is more of a problem in politics and the media than it is in the normal practice of engineering and science, it seems useful to touch on it brießy here since the underlying issue is a consequence of scale.

We start by reconsidering some calculations we have already performed (in Section 3.5.2) to show how we can use a scale factor to effectively move

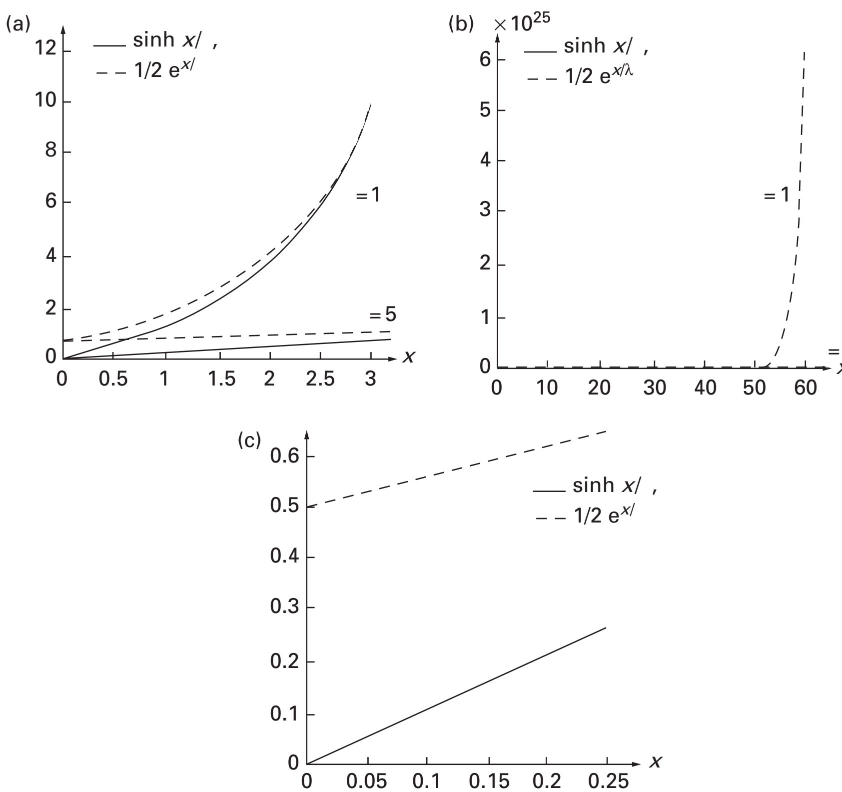

Figure 3.15 Plots of sinh (x/λ) (solid line) and its one-term exponential approximation (1/2) exp (x/λ) (dashed line): (a) for $\lambda = 1, 5$, with length scale $0 \le x \le 3.0$; (b) for $\lambda = 1, 5$, with length scale $0 \le x \le 60$; and (c) for $\lambda = 1$, with length scale $0 \le x \le 0.25$.

a boundary. In Figure 3.15(a) we display plots of $\sinh(x/\lambda)$ (solid line) and its one-term exponential approximation (dashed line). We now see what we have previously described, namely, for $\lambda = 1$ the approximation is good for $x \geq 3$, while for $\lambda = 5$ the approximation works for $x \geq 15$. The scale factor λ makes the approximation valid for different values of x because of the argument that $e^{-x/\lambda}$ can be neglected when compared to 1 for $x/\lambda \geq 3$.

The same two functions have been redrawn in Figure 3.15(b) where the horizontal scale has been very much contracted, as a result of which we donÕt see any difference between the hyberbolic sinusoid and its elementary approximation. That is, it looks like $\sinh(x/\lambda)$ and $1/2e^{x/\lambda}$ are the same for all values of x, when we know that is not the case. In other words, we have lost (or hidden) some information about the behavior at small values of x. To emphasize this, we show in Figure 3.15(c) a plot for the case $\lambda = 1$ with a much-elongated horizontal scale where, as a result, the difference between the two functions is very much exaggerated.

Lastly on graphical display, scaling, and perception, we show in Figures 3.16 and 3.17 two illustrations of the consequences of scale in contexts somewhat beyond the normal professional concerns of engineers and scientists. We show both examples because they use the same technique of carefully choosing a scale in a Þgure in order to present data out of context. In Figure 3.16(a) we show a rather dated picture of trafÞc deaths in the state of Connecticut during the time interval 1956Ð1957, and we see that a sharp drop in trafÞc deaths occurred then. But, was that drop real? And, in comparison to what? It turns out that if more data are added, as in Figure 3.16(b), we see that the drop followed a rather precipitous increase in the number of trafÞc fatalities. Further, if we added data from adjacent states and normalized the number of deaths against a common base, as shown in Figure 3.16(c), we then Þnd that the numbers of ConnecticutÕs trafÞc fatalities was similar to those of its neighbors, although the impact of the stricter enforcement is still visible after 1955.

Similarly, one of the most often shown graphics in the Þnancial pages of newspapers, or in their televised equivalents, are graphics such as that shown in Figure 3.17 (see p. 65). Here, the immediate sense conveyed is that the bottom has dropped out of the market because the scale used on the ordinate (or y- or vertical axis) has been so foreshortened that it includes only one weekÕs trading activities, Thus, a decline of a few percent in a stock market barometer such as the Dow Jones Industrial Average (DJIA) is made to look like a much steeper declineÑ especially if the curve itself is drawn in red ink!

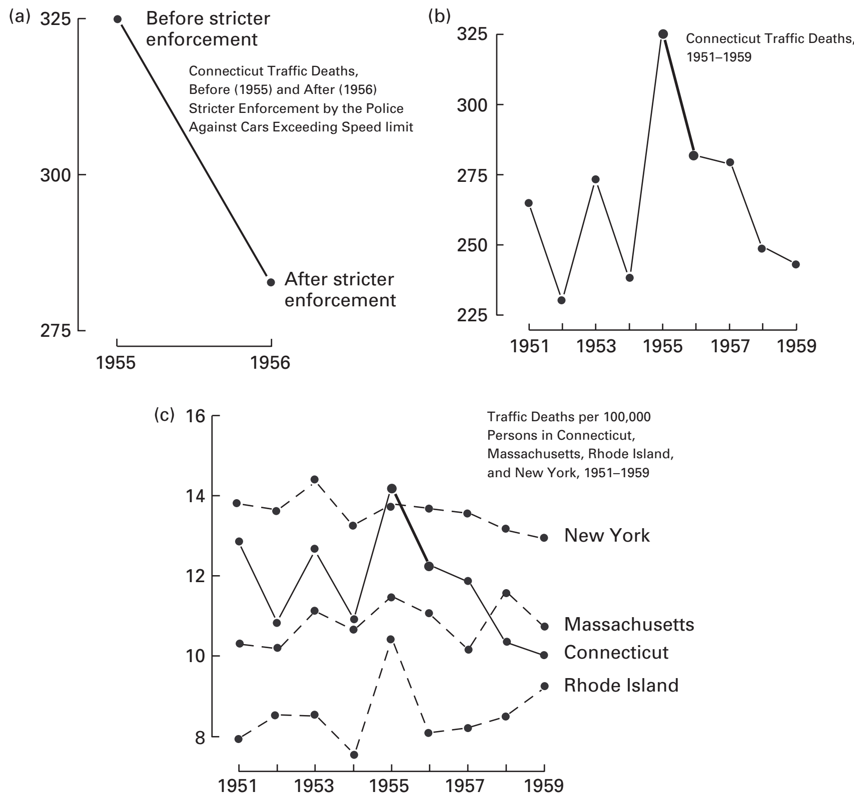

Figure 3.16 Plots of traffic fatalities in the state of Connecticut, showing the dangers of truncating scales and deleting comparative data: (a) Connecticut data for 1955–56; (b) Connecticut data for 1951–59; and (c) normalized data for Connecticut and three neighboring states for 1951–59 (from Tufte, 1983).

Problem 3.19. Show that the deßection w_{mp} of a beam with bending stiffness *EI*, length *L*, and under a concentrated load *P* is governed by the two dimensionless groups in eq. (3.34).

Problem 3.20. Why is the torque, τ, in the apparatus of Figure 3.12 a constant?

Problem 3.21. Expressed in terms of the wheel© geometric and gravitational properties, what is the magnitude of the torque in Problem 3.20?

Problem 3.22. ConÞrm that eq. (3.30) is dimensionally correct.

Problem 3.23. ConÞrm that eq. (3.31) is dimensionally correct.

Problem 3.24. Calculate and conÞrm the estimated number of revolutions in the last column of Table 3.1.

Problem 3.25. ConÞrm that eq. (3.39) is the correct representation of eq. (3.37) in terms of the Þve scaling factors of a simple beam.

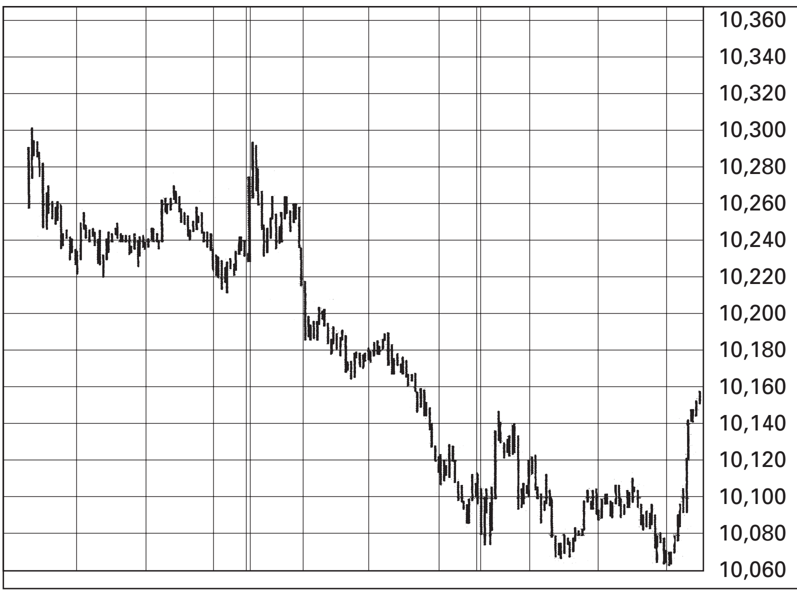

Figure 3.17 A plot of the performance of the New York Stock Exchange during 13–15 May 2002, as exemplified by that universally-cited barometer, the Dow Jones Industrial Average (DJIA) (www.bigcharts.com, 2002).

3.7 Summary

Continuing the discussion of issues involving dimensions that began in Chapter 2, here we have focused on very important effects of scale. We have shown how scaling effects inßuenced the growth of cathedrals and large churches, and we have demonstrated how size affects function in the ability of birds to hover and in peopleÕs ability to hear and to speak. In fact, we have shown that the nature of hearing and speech in animals is determined in large part by the relative size of the relevant parts of their anatomy.

We have also discussed the fact that scaling has a signiÞcant effect on experiments, both in terms of how data is acquired and how it is interpreted. The choice of scale(s) for experiments is a crucial part of the design of experiments. More generally, we have seen that the way that data is scaled for presentation can signiÞcantly inßuence how people perceive the meaning of that data. This is also a very important part of modeling because it speaks directly to the perceived credibility of the results of any modeling endeavor.

3.8 References

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3.9 Problems

3.26. Formulate a hypothesis to explain why a wood pigeon and a buzzard seem to have such different ratios of W_{fm}/W_b in Figure 3.2.

3.27. Show that the equation that describes the log-log plot of Figure 3.7 can be found to be $h \cong 1.23\, l^{0.68}$, where h and l are, respectively, the nave height and church length rendered dimensionless by dividing each by 1 ft.

3.28. Using reasoning similar to that which brought us to eq. (3.13), show that the maximum speed at which animals can run is independent of size.

3.29. The velocity of blood in the aorta is related to the difference in pressure between the heart and the arteries. Find the relationship between the velocity of the blood and the pressure difference. (*Hint*: Use the work-energy theorem as we did for bird hovering in Section 3.3.2.)

3.30. The stilt, a little long-legged bird, was described in *Gulliver's Travels* as weighing 4.5 ounces and having legs that are 8 in long. A ßamingo has a similar shape and weighs 4 lb. Apply scaling arguments to show that ßamingoÕlegs should be about 20 in long (as they actually are!).

3.31. Given that a robin weighs about 2 ounces, could we scale the length of its legs from the stilt data given in Problem 3.30? Explain your answer.

3.32. A certain cucumber was found to have cells that divided when they had grown to 1.5 times the volume of ÒrestingÓcells. Cells normally divide so that the ratio between their surface and their mass remains constant. Is the cucumber described a ÒnormalÓcucumber?

3.33. Find the range of values of the variable x for which the approximation

$$\cosh(x/\lambda) \cong \frac{1}{2} e^{x/\lambda}$$

is acceptable, for scaling factors $\lambda = 1$ and $\lambda = 6$. Plot both functions for each of the two scaling factors.

3.34. An experiment to determine the natural or fundamental period of oscillation of a simple spring-mass system (see Figure P3.34) is set up as follows. A spring of stiffness k is Þxed at one end and connected to a mass m at its other, with the mass being able to move along an ideal, frictionless air track. The mass is displaced a distance x_0 from its initial resting position, after which it oscillates along the air track around that initial position. The time needed for a complete oscillationÑ that is, the periodÑ is measured several times for several periods in succession, with the results being compared to the theoretical formula for the period, T:

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

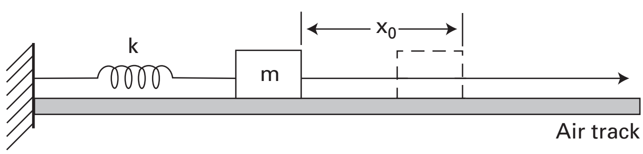

Figure P3.34 An experimental device for determining the period, T, of a spring-mass system, wherein the mass, m, moves on an ideal, frictionless air track.

Assuming that k and m are known, and that the timer used to measure the period is accurate to within $\pm 1\%$. What are the possible pitfalls that could prevent the successful experimental determination of T?

3.35. When the structural elements called *beams* vibrate freely, their natural frequencies, ω, depend on a beamÕmass density, ρ, its modulus of elasticity, E, and its length, l, depth, h, and cross-sectional area, A. If a model and prototype are to be built of the same material and tested, and their lengths are scaled in the ratio 1:5, how will their natural frequencies relate? (*Hint*: Use dimensional analysis to determine the various dimensionless parameters that relate ω to the various beam properties.)

3.36. A steel beam of length of 20 cm is to be used to model a prototype timber beam whose span is 3.6 m.

(a) Verify that the dimensionless group containing the load, the modulus, and the length is P/EL^2.

(b) If the timber beam is to carry a load of 9000 N at a point 1.5 m from the left end, what load must be applied to the model to determine whether the prototype can carry its intended load? (Assume that the load-carrying capacity is the only behavior of interest here.)

3.37. The data given in the table immediately below were recorded as the growth of a colony of bacteria was observed. (a) Plot this data as a function of time. (b) Write an equation that expresses the bacterial population as a function of time.

Time (min)	Population (p) $\times 10^6$
0	10
5	15
10	22
20	50
30	110
40	245
50	546
60	1,215
70	2,704
80	6,018
90	13,394
100	29,810