



2

Dimensional Analysis

We begin this chapter, the first of three dealing with the *tools* or techniques for mathematical modeling, with H. L. Langhaar's definition of *dimensional analysis*:

Dimensional analysis (*n*): a method by which we deduce information about a phenomenon from the single premise that the phenomenon can be described by a dimensionally correct equation among certain variables.

This quote expresses the simple, yet powerful idea that we introduced in Section 1.3.1: all of the terms in our equations must be *dimensionally consistent*, that is, each separate term in those equations must have the same net physical dimensions. For example, when summing forces to ensure equilibrium, every term in the equation must have the physical dimension of force. (Equations that are dimensionally consistent are sometimes called *rational*.) This idea is particularly useful for validating newly developed mathematical models or for confirming formulas and equations before doing calculations with them. However, it is also a weak statement because the available tools of dimensional analysis are rather limited, and applying them does not always produce desirable results.

2.1 Dimensions and Units

The physical quantities we use to model objects or systems represent *concepts*, such as time, length, and mass, to which we also attach *numerical* values or measurements. Thus, we could describe the width of a soccer field by saying that it is 60 meters wide. The concept or abstraction invoked is length or distance, and the numerical measure is 60 meters. The numerical measure implies a comparison with a standard that enables both communication about and comparison of objects or phenomena without their being in the same place. In other words, common measures provide a frame of reference for making comparisons. Thus, soccer fields are wider than American football fields since the latter are only 49 meters wide.

The physical quantities used to describe or model a problem come in two varieties. They are either *fundamental* or *primary* quantities, or they are *derived* quantities. Taking a quantity as fundamental means only that we can assign it a standard of measurement independent of that chosen for the other fundamental quantities. In mechanical problems, for example, mass, length, and time are generally taken as the fundamental mechanical variables, while force is derived from Newton's law of motion. It is equally correct to take force, length, and time as fundamental, and to derive mass from Newton's law. For any given problem, of course, we need enough fundamental quantities to express each derived quantity in terms of these primary quantities.

While we relate primary quantities to standards, we also note that they are chosen arbitrarily, while derived quantities are chosen to satisfy physical laws or relevant definitions. For example, length and time are fundamental quantities in mechanics problems, and speed is a derived quantity expressed as length per unit time. If we chose time and speed as primary quantities, the derived quantity of length would be (speed \times time), and the derived quantity of area would be (speed \times time)².

The word *dimension* is used to relate a derived quantity to the fundamental quantities selected for a particular model. If mass, length, and time are chosen as primary quantities, then the dimensions of area are (length)², of mass density are mass/(length)³, and of force are (mass \times length)/(time)². We also introduce the notation of brackets [] to read as "the dimensions of." If **M**, **L**, and **T** stand for mass, length, and time, respectively, then:

$$[A = \text{area}] = (\text{L})^2, \quad (2.1a)$$

$$[\rho = \text{density}] = \text{M}/(\text{L})^3, \quad (2.1b)$$

$$[F = \text{force}] = (\text{M} \times \text{L})/(\text{T})^2. \quad (2.1c)$$

The *units* of a quantity are the numerical aspects of a quantity's dimensions expressed in terms of a given physical standard. By definition, then, a unit is an arbitrary multiple or fraction of that standard. The most widely accepted international standard for measuring length is the meter (m), but it can also be measured in units of centimeters (1 cm = 0.01 m) or of feet (0.3048 m). The magnitude or size of the attached number obviously depends on the unit chosen, and this dependence often suggests a choice of units to facilitate calculation or communication. The soccer field width can be said to be 60 m, 6000 cm, or (approximately) 197 feet.

Dimensions and units are related by the fact that identifying a quantity's dimensions allows us to compute its numerical measures in different sets of units, as we just did for the soccer field width. Since the physical dimensions of a quantity are the same, there must exist numerical relationships between the different systems of units used to measure the amounts of that quantity. For example,

$$1 \text{ foot (ft)} \cong 30.48 \text{ centimeters (cm)},$$

$$1 \text{ centimeter (cm)} \cong 0.000006214 \text{ miles (mi)},$$

$$1 \text{ hour (hr)} = 60 \text{ minutes (min)} = 3600 \text{ seconds (sec or s)}.$$

This equality of units for a given dimension allows us to change or convert units with a straightforward calculation. For a speed of 65 miles per hour (mph), for example, we can calculate the following equivalent:

$$65 \frac{\text{mi}}{\text{hr}} = 65 \frac{\text{mi}}{\text{hr}} \times 5280 \frac{\text{ft}}{\text{mi}} \times 0.3048 \frac{\text{m}}{\text{ft}} \times 0.001 \frac{\text{km}}{\text{m}} \cong 104.6 \frac{\text{km}}{\text{hr}}.$$

Each of the multipliers in this conversion equation has an effective value of unity because of the equivalencies of the various units, that is, 1 mi = 5280 ft, and so on. This, in turn, follows from the fact that the numerator and denominator of each of the above multipliers have the same physical dimensions. We will discuss *systems of units* and provide some conversion data in Section 2.4.

2.2 Dimensional Homogeneity

We had previously defined a *rational equation* as an equation in which each independent term has the same net dimensions. Then, taken in its entirety, the equation is *dimensionally homogeneous*. Simply put, we cannot add length to area in the same equation, or mass to time, or charge to stiffness—although we can add (and with great care) quantities having the same dimensions but expressed in different units, e.g., length in meters

and length in feet. The fact that equations must be rational in terms of their dimensions is central to modeling because it is one of the best—and easiest—checks to make to determine whether a model makes sense, has been correctly derived, or even correctly copied!

We should remember that a dimensionally homogeneous equation is independent of the units of measurement being used. However, we can create unit-dependent versions of such equations because they may be more convenient for doing repeated calculations or as a memory aid. In an example familiar from mechanics, the period (or cycle time), T_0 , of a pendulum undergoing small-angle oscillations can be written in terms of the pendulum's length, l , and the acceleration of gravity, g :

$$T_0 = 2\pi\sqrt{l/g}. \quad (2.2)$$

This dimensionally homogeneous equation is independent of the system of units chosen to measure length and time. On the other hand, we may find it convenient to work in the metric system, in which case $g = 9.8 \text{ m/s}^2$, from which it follows that

$$T_0(\text{s}) = 2\pi\sqrt{l/9.8} \cong 2\sqrt{l}. \quad (2.3)$$

Equation (2.3) is valid *only* when the pendulum's length is measured in meters. In the so-called British system,¹ where $g = 32.17 \text{ ft/sec}^2$,

$$T_0(\text{sec}) = 2\pi\sqrt{l/32.17} \cong 1.1\sqrt{l}. \quad (2.4)$$

Why not?

Equations (2.3) and (2.4) are not dimensionally homogeneous. So, while these formulas may be appealing or elegant, we have to remember their limited ranges of validity, as we should whenever we use or create similar formulas for whatever modeling we are doing.

2.3 Why Do We Do Dimensional Analysis?

We presented a definition of dimensional analysis at the beginning of this chapter, where we also noted that the “method” so defined has both powerful implications—rational equations and dimensional consistency—and severe limitations—the limited nature of the available tools. Given this limitation, why has this method or technique developed, and why has it persisted?

¹ One of my Harvey Mudd colleagues puckishly suggests that we should call this the American system of units as we are, apparently, the only country still so attached to feet and pounds.



Figure 2.1 A picture of a “precision mix batch mixer” that would be used to mix large quantities of foods such as peanut butter and other mixes of substances that have relatively high values of density ρ and viscosity μ (courtesy of H. C. Davis Sons Manufacturing Company, Inc.).

Dimensional analysis developed as an attempt to perform extended, costly experiments in a more organized, more efficient fashion. The underlying idea was to see whether the number of variables could be grouped together so that fewer trial runs or fewer measurements would be needed. Dimensional analysis produces a more compact set of outputs or data, with perhaps fewer charts and graphs, which in turn might better clarify what is being observed.

Imagine for a moment that we want to design a machine to make large quantities of peanut butter (and this author prefers creamy to crunchy!). We can imagine a mixer that takes all of the ingredients (i.e., roasted peanuts, sugar, and “less than 2%” of molasses and partially hydrogenated vegetable oil) and mixes them into a smooth, creamy spread. Moving a knife through a jar of peanut butter requires a noticeably larger force than stirring a glass of water. Similarly, the forces in a vat-like mixer would be considerable, as would the power needed to run that mixer in an automated food assembly line, as illustrated in Figure 2.1. Thus, the electro-mechanical design of an industrial-strength peanut butter mixer depends on estimates of the forces required to mix the peanut butter. How can we get some idea of what those forces are?

It turns out, as you might expect, that the forces depend in large part on properties of the peanut butter, but on which properties, and how? We can

answer those questions by performing a series of experiments in which we push a blade through a tub of peanut butter and measure the amount of force required to move the blade at different speeds. We will call the force needed to move the blade through the peanut butter the drag force, F_D , because it is equal to the force exerted by the moving (relatively speaking) peanut butter to retard the movement of the knife. We postulate that the force depends on the speed V with which the blade moves, on a characteristic dimension of the blade, say the width d , and on two characteristics of the peanut butter. One of these characteristics is the mass density, ρ , and the second is a parameter called the *viscosity*, μ , which is a measure of its “stickiness.” If we think about our experiences with various fluids, including water, honey, motor oil, and peanut butter, these two characteristics seem intuitively reasonable because we do associate a difficulty in stirring (and cleaning up) with fluids that feel heavier and stickier.

Thus, the five quantities that we will take as derived quantities for this initial investigation into the mixing properties of peanut butter are the drag force, F_D , the speed with which the blade moves, V , the knife blade width, d , the peanut butter’s mass density, ρ , and its viscosity, μ . The fundamental physical quantities we would apply here are mass, length, and time, which we denote as **M**, **L**, and **T**, respectively. The derived variables are expressed in terms of the fundamental quantities in Table 2.1.

Table 2.1 The five derived quantities chosen to model the peanut butter stirring experiments.

Derived quantities	Dimensions
Speed (V)	L/T
Blade width (d)	L
Density (ρ)	$M/(L)^3$
Viscosity (μ)	$M/(L \times T)$
Drag force (F_D)	$(M \times L)/(T)^2$

How did we get the fundamental dimensions of the viscosity? By a straightforward application of the principle of dimensional homogeneity to the assumptions used in modeling the mechanics of fluids: The drag force (or force required to pull the blade through the butter) is directly proportional both to the speed with which it moves and the area of the blade, and inversely proportional to a length that characterizes the spatial rate of change of the speed. Thus,

$$F_D \propto \frac{VA}{L}, \quad (2.5a)$$

or

$$F_D = \mu \frac{VA}{L}. \quad (2.5b)$$

If we apply the principle of dimensional homogeneity to eq. (2.5b), it follows that

$$[\mu] = \left[\frac{F_D}{A} \times \frac{L}{V} \right]. \quad (2.6)$$

It is easy to show that eq. (2.6) leads to the corresponding entry in Table 2.1.

Now we consider the fact that we want to know how F_D and V are related, and yet they are also functions of the other variables, d , ρ , and μ , that is,

$$F_D = F_D(V; d, \rho, \mu). \quad (2.7)$$

Equation (2.7) suggests that we would have to do a lot of experiments and plot a lot of curves to find out how drag force and speed relate to each other while we are also varying the blade width and the butter density and viscosity. If we wanted to look at only three different values of each of d , ρ , and μ , we would have nine (9) different graphs, each containing three (3) curves. This is a significant accumulation of data (and work!) for a relatively simple problem, and it provides a very graphic illustration of the need for dimensional analysis. We will soon show that this problem can be “reduced” to considering two dimensionless groups that are related by a single curve! Dimensional analysis is thus very useful for both designing and conducting experiments.

Problem 2.1. Justify the assertion made just above that “nine (9) different graphs, each containing three (3) curves” are needed to relate force and speed.

Problem 2.2. Find and compare the mass density and viscosity of peanut butter, honey, and water.

2.4 How Do We Do Dimensional Analysis?

Dimensional analysis is the process by which we ensure dimensional consistency. It ensures that we are using the proper dimensions to describe the problem being modeled, whether expressed in terms of the correct number of properly dimensioned variables and parameters or whether written in terms of appropriate dimensional groups. Remember, too, that we need consistent dimensions for logical consistency, and we need consistent units for arithmetic consistency.

How do we ensure dimensional consistency? First, we check the dimensions of all derived quantities to see that they are properly represented in terms of the chosen primary quantities and their dimensions. Then we identify the proper *dimensionless groups* of variables, that is, ratios and products of problem variables and parameters that are themselves dimensionless. We will explain two different techniques for identifying such dimensionless groups, the *basic method* and the *Buckingham Pi theorem*.

2.4.1 The Basic Method of Dimensional Analysis

The basic method of dimensional analysis is a rather informal, unstructured approach for determining dimensional groups. It depends on being able to construct a functional equation that contains all of the relevant variables, for which we know the dimensions. The proper dimensionless groups are then identified by the thoughtful elimination of dimensions.

For example, consider one of the classic problems of elementary mechanics, the free fall of a body in a vacuum. We recall that the speed, V , of such a falling body is related to the gravitational acceleration, g , and the height, h , from which the body was released. Thus, the functional expression of this knowledge is:

$$V = V(g, h). \quad (2.8)$$

Note that the precise form of this functional equation is, at this point, entirely unknown—and we don't need to know that form for what we're doing now. The physical dimensions of the three variables are:

$$\begin{aligned} [V] &= \text{L}/\text{T}, \\ [g] &= \text{L}/\text{T}^2, \\ [h] &= \text{L}. \end{aligned} \quad (2.9)$$

The time dimension, T , appears only in the speed and gravitational acceleration, so that dividing the speed by the square root of g eliminates time and yields a quantity whose remaining dimension can be expressed entirely in terms of length, that is:

$$\left[\frac{V}{\sqrt{g}} \right] = \sqrt{\text{L}}. \quad (2.10)$$

If we repeat this thought process with regard to eliminating the length dimension, we would divide eq. (2.10) by \sqrt{h} , which means that

$$\left[\frac{V}{\sqrt{gh}} \right] = 1. \quad (2.11)$$

Since we have but a single dimensionless group here, it follows that:

$$V = \text{constant} \times (\sqrt{gh}) \quad (2.12)$$

Thus, the speed of a falling body is proportional to \sqrt{gh} , a result we should recall from physics—yet we have found it with dimensional analysis alone, without invoking Newton’s law or any other principle of mechanics. This elementary application of dimensional consistency tells us something about the power of dimensional analysis. On the other hand, we do need some physics, either theory or experiment, to define the constant in eq. (2.12).

Someone seeing the result (2.12) might well wonder why the speed of a falling object is independent of mass (unless that person knew of Galileo Galilei’s famous experiment). In fact, we can use the basic method to build on eq. (2.12) and show why this is so. Simply put, we start with a functional equation that included mass, that is,

$$V = V(g, h, m). \quad (2.13)$$

A straightforward inspection of the dimensions of the four variables in eq. (2.13), such as the list in eq. (2.9), would suggest that mass is not a variable in this problem because it only occurs once as a dimension, so it cannot be used to make eq. (2.13) dimensionless.

As a further illustration of the basic method, consider the mutual revolution of two bodies in space that is caused by their mutual gravitational attraction. We would like to find a dimensionless function that relates the period of oscillation, T_R , to the two masses and the distance r between them:

$$T_R = T_R(m_1, m_2, r). \quad (2.14)$$

If we list the dimensions for the four variables in eq. (2.14) we find:

$$\begin{aligned} [m_1], [m_2] &= \mathbf{M}, \\ [T_R] &= \mathbf{T}, \\ [r] &= \mathbf{L}. \end{aligned} \quad (2.15)$$

We now have the converse of the problem we had with the falling body. Here none of the dimensions are repeated, save for the two masses. So, while we can expect that the masses will appear in a dimensionless ratio, how do we keep the period and distance in the problem? The answer is that we need to add a variable containing the dimensions heretofore missing to

22 Chapter 2 Dimensional Analysis

the functional equation (2.14). Newton’s gravitational constant, G , is such a variable, so we restate our functional equation (2.14) as

$$T_R = T_R(m_1, m_2, r, G), \tag{2.16}$$

where the dimensions of G are

$$[G] = \text{L}^3/\text{MT}^2. \tag{2.17}$$

The complete list of variables for this problem, consisting of eqs. (2.15) and (2.17), includes enough variables to account for all of the dimensions.

Regarding eq. (2.16) as the correct functional equation for the two revolving bodies, we apply the basic method first to eliminate the dimension of time, which appears directly in the period T_R and as a reciprocal squared in the gravitational constant G . It follows dimensionally that

$$\left[T_R \sqrt{G} \right] = \sqrt{\frac{\text{L}^3}{\text{M}}}, \tag{2.18a}$$

where the right-hand side of eq. (2.18a) is independent of time. Thus, the corresponding revised functional equation for the period would be:

$$T_R \sqrt{G} = T_{R1}(m_1, m_2, r). \tag{2.18b}$$

We can eliminate the length dimension simply by noting that

$$\left[\frac{T_R \sqrt{G}}{\sqrt{r^3}} \right] = \sqrt{\frac{1}{\text{M}}}, \tag{2.19a}$$

which leads to a further revised functional equation,

$$\frac{T_R \sqrt{G}}{\sqrt{r^3}} = T_{R2}(m_1, m_2). \tag{2.19b}$$

We see from eq. (2.19a) that we can eliminate the mass dimension from eq. (2.19b) by multiplying eq. (2.19b) by the square root of one of the two masses. We choose the square root of the second mass (do Problem 2.6 to find out what happens if the first mass is chosen), $\sqrt{m_2}$, and we find from eq. (2.19a) that

$$\left[\frac{T \sqrt{G m_2}}{\sqrt{r^3}} \right] = 1. \tag{2.20a}$$

This means that eq. (2.19b) becomes

$$\frac{T_R \sqrt{G m_2}}{\sqrt{r^3}} = \sqrt{m_2} T_{R2}(m_1, m_2) \equiv T_{R3} \left(\frac{m_1}{m_2} \right), \tag{2.20b}$$

where a dimensionless mass ratio has been introduced in eq. (2.20b) to recognize that this is the only way that the function T_{R3} can be both dimensionless *and* a function of the two masses. Thus, we can conclude from eq. (2.20b) that

$$T_R = \sqrt{\frac{r^3}{Gm_2}} T_{R3} \left(\frac{m_1}{m_2} \right). \quad (2.21)$$

This example shows that difficulties arise if we start a problem with an incomplete set of variables. Recall that we did not include the gravitational constant G until it became clear that we were headed down a wrong path. We then included G to rectify an incomplete analysis. With the benefit of hindsight, we might have argued that the attractive gravitational force must somehow be accounted for, and including G could have been a way to do that. This argument, however, demands insight and judgment whose origins may have little to do with the particular problem at hand.

While our applications of the basic method of dimensional analysis show that it does not have a formal algorithmic structure, it can be described as a series of steps to take:

- a. List all of the variables and parameters of the problem and their dimensions.
- b. Anticipate how each variable qualitatively affects quantities of interest, that is, does an increase in a variable cause an increase or a decrease?
- c. Identify one variable as depending on the remaining variables and parameters.
- d. Express that dependence in a functional equation (i.e., analogs of eqs. (2.8) and (2.14)).
- e. Choose and then eliminate one of the primary dimensions to obtain a revised functional equation.
- f. Repeat steps (e) until a revised, *dimensionless* functional equation is found.
- g. Review the final *dimensionless* functional equation to see whether the apparent behavior accords with the behavior anticipated in step “b”.

Problem 2.3. What is the constant in eq. (2.12)? How do you know that?

Problem 2.4. Apply the basic method to eq. (2.2) for the period of the pendulum.

Problem 2.5. Carry out the basic method for eq. (2.13) and show that the mass of a falling body does not affect its speed of descent.

Problem 2.6. Carry out the last step of the basic method for eqs. (2.20) using the first mass and show it produces a form that is equivalent to eq. (2.21).

2.4.2 The Buckingham Pi Theorem for Dimensional Analysis

Buckingham's Pi theorem, fundamental to dimensional analysis, can be stated as follows:

A dimensionally homogeneous equation involving n variables in m primary or fundamental dimensions can be reduced to a single relationship among $n - m$ independent dimensionless products.

A dimensionally homogeneous (or rational) equation is one in which every independent, additive term in the equation has the same dimensions. This means that we can solve for any one term as a function of all the others. If we introduce Buckingham's Π notation to represent a dimensionless term, his famous Pi theorem can be written as:

$$\Pi_1 = \Phi(\Pi_2, \Pi_3 \dots \Pi_{n-m}). \quad (2.22a)$$

or, equivalently,

$$\Phi(\Pi_1, \Pi_2, \Pi_3 \dots \Pi_{n-m}) = 0. \quad (2.22b)$$

Equations (2.22) state that a problem with n derived variables and m primary dimensions or variables requires $n - m$ dimensionless groups to correlate all of its variables.

We apply the Pi theorem by first identifying the n derived variables in a problem: $A_1, A_2, \dots A_n$. We choose m of these derived variables such that they contain all of the m primary dimensions, say, A_1, A_2, A_3 for $m = 3$. Dimensionless groups are then formed by permuting each of the remaining $n - m$ variables ($A_4, A_5, \dots A_n$ for $m = 3$) in turn with those m 's already chosen:

$$\begin{aligned} \Pi_1 &= A_1^{a_1} A_2^{b_1} A_3^{c_1} A_4, \\ \Pi_2 &= A_1^{a_2} A_2^{b_2} A_3^{c_2} A_5, \\ &\vdots \\ \Pi_{n-m} &= A_1^{a_{n-m}} A_2^{b_{n-m}} A_3^{c_{n-m}} A_n. \end{aligned} \quad (2.23)$$

The a_i , b_i , and c_i are chosen to make each of the permuted groups Π_i dimensionless.

For example, for the peanut butter mixer, there should be two dimensionless groups correlating the five variables of the problem (listed in Table 2.1). To apply the Pi theorem to this mixer we choose the blade speed V , its width d , and the butter density ρ as the fundamental variables ($m = 3$), which we then permute with the two remaining variables—the viscosity μ and the drag force F_D —to get two dimensionless groups:

$$\begin{aligned}\Pi_1 &= V^{a_1} d^{b_1} \rho^{c_1} \mu, \\ \Pi_2 &= V^{a_2} d^{b_2} \rho^{c_2} F_D.\end{aligned}\tag{2.24}$$

Expressed in terms of primary dimensions, these groups are:

$$\begin{aligned}\Pi_1 &= \left(\frac{\text{L}}{\text{T}}\right)^{a_1} \text{L}^{b_1} \left(\frac{\text{M}}{\text{L}^3}\right)^{c_1} \left(\frac{\text{M}}{\text{LT}}\right), \\ \Pi_2 &= \left(\frac{\text{L}}{\text{T}}\right)^{a_2} \text{L}^{b_2} \left(\frac{\text{M}}{\text{L}^3}\right)^{c_2} \left(\frac{\text{ML}}{\text{T}^2}\right).\end{aligned}\tag{2.25}$$

Now, in order for Π_1 and Π_2 to be dimensionless, the net exponents for each of the three primary dimensions must vanish. Thus, for Π_1 ,

$$\begin{aligned}\text{L} : & a_1 + b_1 - 3c_1 - 1 = 0, \\ \text{T} : & -a_1 - 1 = 0, \\ \text{M} : & c_1 + 1 = 0,\end{aligned}\tag{2.26a}$$

and for Π_2 ,

$$\begin{aligned}\text{L} : & a_2 + b_2 - 3c_2 + 1 = 0, \\ \text{T} : & -a_2 - 2 = 0, \\ \text{M} : & c_2 + 1 = 0.\end{aligned}\tag{2.26b}$$

Solving eqs. (2.26) for the two pairs of subscripts yields:

$$\begin{aligned}a_1 = b_1 = c_1 &= -1, \\ a_2 = b_2 = -2, \quad c_2 &= -1.\end{aligned}\tag{2.27}$$

Then the two dimensionless groups for the peanut butter mixer are:

$$\begin{aligned}\Pi_1 &= \left(\frac{\mu}{\rho V d}\right), \\ \Pi_2 &= \left(\frac{F_D}{\rho V^2 d^2}\right).\end{aligned}\tag{2.28}$$

Thus, there are two dimensionless groups that should guide experiments with prototype peanut butter mixers. One clearly involves the viscosity of the peanut butter, while the other relates the drag force on the blade to

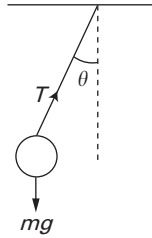


Figure 2.2 The classical pendulum oscillating through angles θ due to gravitational acceleration g .

the blade's dimensions and speed, as well as to the density of the peanut butter.

In Chapter 7 we will explore one of the “golden oldies” of physics, modeling the small angle, free vibration of an ideal pendulum (viz. Figure 2.2). There are six variables to consider in this problem, and they are listed along with their fundamental dimensions in Table 2.2. In this case we have $m = 6$ and $n = 3$, so that we can expect three dimensionless groups. We will choose l , g , and m as the variables around which we will permute the remaining three variables (T_0 , θ , T) to obtain the three groups. Thus,

$$\begin{aligned}\Pi_1 &= l^{a_1} g^{b_1} m^{c_1} T_0, \\ \Pi_2 &= l^{a_2} g^{b_2} m^{c_2} \theta, \\ \Pi_3 &= l^{a_3} g^{b_3} m^{c_3} T.\end{aligned}\tag{2.29}$$

Table 2.2 The six derived quantities chosen to model the oscillating pendulum.

Derived quantities	Dimensions
Length (l)	L
Gravitational acceleration (g)	L/T ²
Mass (m)	M
Period (T_0)	T
Angle (θ)	1
String tension (T)	(M × L)/T ²

The Pi theorem applied here then yields three dimensionless groups (see Problem 2.9):

$$\begin{aligned}\Pi_1 &= \frac{T_0}{\sqrt{l/g}}, \\ \Pi_2 &= \theta, \\ \Pi_3 &= \frac{T}{mg}.\end{aligned}\tag{2.30}$$

These groups show how the period depends on the pendulum length l and the gravitational constant g (recall eq. (2.2)), and the string tension T on the mass m and g . The second group also shows that the (dimensionless) angle of rotation stands alone, that is, it is apparently not related to any of the other variables. This follows from the assumption of small angles, which makes the problem linear, and makes the magnitude of the angle of free vibration a quantity that cannot be determined.

One of the “rules” of applying the Pi theorem is that the m chosen variables include all n of the fundamental dimensions, but no other restrictions are given. So, it is natural to ask how this analysis would change if we start with three different variables. For example, suppose we choose T_0 , g , and m as the variables around which to permute the remaining three variables (l , θ , T) to obtain the three groups. In this case we would write:

$$\begin{aligned}\Pi'_1 &= T_0^{a_1} g^{b_1} m^{c_1} l, \\ \Pi'_2 &= T_0^{a_2} g^{b_2} m^{c_2} \theta, \\ \Pi'_3 &= T_0^{a_3} g^{b_3} m^{c_3} T.\end{aligned}\tag{2.31}$$

Applying the Pi theorem to eq. (2.31) yields the following three “new” dimensionless groups (see Problem 2.10):

$$\begin{aligned}\Pi'_1 &= \frac{l/g}{T_0^2} = \frac{1}{\Pi_1^2}, \\ \Pi'_2 &= \theta = \Pi_2, \\ \Pi'_3 &= \frac{T}{mg} = \Pi_3.\end{aligned}\tag{2.32}$$

We see that eq. (2.32) produce the same information as eq. (2.30), albeit in a slightly different form. In particular, it is clear that Π_1 and Π'_1 contain the same dimensionless group, which suggests that the number of dimensionless groups is unique, but that the precise forms that these groups

may take are not. This last calculation demonstrates that the dimensionless groups determined in any one calculation are unique in one sense, but they may take on different, yet related forms when done in a slightly different calculation.

Note that our applications of the basic method and the Buckingham Pi theorem of dimensional analysis can be cast in similar, step-like structures. However, experience and insight are key to applying both methods, even for elementary problems. Perhaps this is a context where, as was said by noted British economist John Maynard Keynes in his famous book, *The General Theory of Employment, Interest, and Money*, “Nothing is required and nothing will avail, except a little, a very little, clear thinking.”

-
- Problem 2.7.** Write out the Buckingham Pi theorem as a series of steps, analogous to the steps described in the basic method on p. 23.
- Problem 2.8.** Confirm eq. (2.28) by applying the basic method to the mixer problem.
- Problem 2.9.** Confirm eq. (2.30) by applying the rest of the Buckingham Pi theorem to the pendulum problem.
- Problem 2.10.** Confirm eq. (2.32) by applying the rest of the Buckingham Pi theorem to the revised formulation of the pendulum problem.
- Problem 2.11.** Apply the Buckingham Pi theorem to the revolution of two bodies in space, beginning with the functional equation (2.16).
- Problem 2.12.** What happens when the Pi theorem is applied to the two-body problem, but beginning now with the functional equation (2.14)?
-

2.5 Systems of Units

We have already noted that units are numerical measures derived from standards. Thus, units are fractions or multiples of those standards. The *British* system has long been the most commonly used system of units in the United States. In the British system, length is typically referenced in *feet*, force in *pounds*, time in *seconds*, and mass in *slugs*. The unit of *pound force* is defined as that force that imparts an acceleration of 32.1740 ft/sec^2 to a mass of $1/2.2046$ of that piece of platinum known as the standard kilogram. While keeping in mind the distinction between dimensions and

Table 2.3 The British and SI systems of units, including abbreviations and (approximate) conversion factors.

Reference Units	British System	SI System
length	foot (ft)	meter (m)
time	second (sec)	second (s)
mass	slug (slug), pound mass (lbm)	kilogram (kg)
force	pound force (lb)	newton (N)
Multiply number of	by	to get
feet (ft)	0.3048	meters (m)
inches (in)	2.540	centimeters (cm)
miles (mi)	1.609	kilometers (km)
miles per hour (mph)	0.447	meters per second (m/s)
pounds force (lb)	4.448	newtons (N)
slugs (slug)	14.59	kilograms (kg)
pounds mass (lbm)	0.454	kilograms (kg)

units, it is worth noting that the fundamental reference quantities in the British system of units (foot, pound, second) are based on the primary dimensions of length (**L**), force (**F**), and time (**T**).

In a belated acknowledgment that the rest of the world (including Britain!) uses *metric* units, a newer system of units is increasingly being adopted in the United States. The *Système International*, commonly identified by its initials, SI, is based on the *mks* system used in physics and it references length in *meters*, mass in *kilograms*, and time in *seconds*. The primary dimensions of the SI system are length (**L**), mass (**M**), and time (**T**). Force is a derived variable in the SI system and it is measured in *newtons*. In Table 2.3 we summarize some of the salient features of the SI and British systems, including the abbreviations used for each unit and the (approximate) conversion factors needed to navigate between the two systems.

Finally, we show in Table 2.4 some of the standard prefixes used to denote the various multiplying factors that are commonly used to denote fractions or multiples of ten (10). To cite a familiar example, in the metric system distances are measured in millimeters (mm), centimeters (cm), meters (m), and kilometers (km).

It is worth noting that some caution is necessary in using the prefixes listed in Table 2.4 because this usage is not universally uniform. For example, the measures used for computer memory are kilobytes (KB), megabytes (MB) and increasingly often these days, gigabytes (GB) and terabytes (TB). The B's stand for bytes. However, the prefixes kilo, mega,

Table 2.4 Some standard numerical factors that are commonly used in the SI system.

Numerical factors (SI)	Prefix (symbol)
10^{-9}	nano (n)
10^{-6}	micro (μ)
10^{-3}	milli (m)
10^{-2}	centi (c)
10^3	kilo (k)
10^6	mega (M)
10^9	giga (G)
10^{12}	tera (T)

giga, and tera, respectively stand for 2^{10} , 2^{20} , 2^{30} , and 2^{40} , which is rather different than what we are using!

To finish off our discussion of numbers we add one final set of approximate equalities, for their interest and possible usefulness:

$$2^{10} \cong e^7 \cong 10^3. \quad (2.33)$$

2.6 Summary

In this chapter we have described an important aspect of problem formulation and modeling, namely, dimensional analysis. Reasoning about the dimensions of a problem requires that we (1) identify all of the physical variables and parameters needed to fully describe a problem, (2) select a set of primary dimensions and variables, and (3) develop the appropriate set of dimensionless groups for that problem. The last step is achieved by applying either the basic method or the more structured Buckingham Pi theorem. The dimensionless groups thus found, along with their numerical values as determined from experiments or further analysis, help us to assess the importance of various effects, to buttress our physical insight and understanding, and to organize our numerical computation, our data gathering, and our design of experiments.

We close by noting that while our use of the formal methods of dimensional analysis will be limited, we will use the concepts of dimensional consistency and dimensional groups extensively. We will see these concepts when we discuss scaling, when we formulate models, and when we solve particular problems. In so doing, we will also keep in mind the distinction

between dimensions and units, and we will also ensure the consistency of units.

2.7 References

- C. L. Dym and E. S. Ivey, *Principles of Mathematical Modeling*, 1st Edition, Academic Press, New York, 1980.
- J. M. Keynes, *The General Theory of Employment, Interest, and Money*, Harcourt, Brace & World, New York, 1964.
- H. L. Langhaar, *Dimensional Analysis and Theory of Models*, John Wiley, New York, 1951.
- H. Schenck, Jr., *Theories of Engineering Experimentation*, McGraw-Hill, New York, 1968.
- R. P. Singh and D. R. Heldman, *Introduction to Food Engineering*, 3rd Edition, Academic Press, San Diego, CA, 2001.
- E. S. Taylor, *Dimensional Analysis for Engineers*, Oxford (Clarendon Press), London and New York, 1974.
- D. F. Young, B. R. Munson, and T. H. Okiishi, *A Brief Introduction to Fluid Mechanics*, 2nd Edition, John Wiley, New York, 2001.

2.8 Problems

- 2.13.** Consider a string of length l that connects a rock of mass m to a fixed point while the rock whirls in a circle at speed v . Use the basic method of dimensional analysis to show that the tension T in the string is determined by the dimensionless group

$$\frac{TL}{mv^2} = \text{constant}.$$

- 2.14.** Apply the Buckingham Pi theorem to confirm the analysis of Problem 2.13.
- 2.15.** The speed of sound in a gas, c , depends on the gas pressure p and on the gas mass density ρ . Use dimensional analysis to determine how c , p , and ρ are related.
- 2.16.** A dimensionless grouping called the Weber number, W_e , is used in fluid mechanics to relate a flowing fluid's surface tension, σ , which has dimensions of force/length, to the fluid's speed, v , density, ρ , and a characteristic length, l . Use dimensional analysis to find that number.

- 2.17.** A pendulum swings in a viscous fluid. How many groups are needed to relate the usual pendulum variables to the fluid viscosity, μ , the fluid mass density, ρ , and the diameter d of the pendulum? Find those groups.
- 2.18.** The volume flow rate Q of fluid through a pipe is thought to depend on the pressure drop per unit length, $\Delta p/l$, the pipe diameter, d , and the fluid viscosity, μ . Use the basic method of dimensional analysis to determine the relation:

$$Q = (\text{constant}) \left(\frac{d^4}{\mu} \right) \left(\frac{\Delta p}{l} \right).$$

- 2.19.** Apply the Buckingham Pi theorem to confirm the analysis of Problem 2.18.
- 2.20.** When flow in a pipe with a rough inner wall (perhaps due to a build-up of mineral deposits) is considered, several variables must be considered, including the fluid speed v , its density ρ and viscosity μ , and the pipe length l and diameter d . The average variation e of the pipe radius can be taken as a measure of the roughness of the pipe's inner surface. Determine the dimensionless groups needed to determine how the pressure drop Δp depends on these variables.
- 2.21.** Use dimensional analysis to determine how the speed of sound in steel depends on the modulus of elasticity, E , and the mass density, ρ . (The modulus of elasticity of steel is, approximately and in British units, 30×10^6 psi.)
- 2.22.** The flexibility (the deflection per unit load) or compliance C of a beam having a square cross-section $d \times d$ depends on the beam's length l , its height and width, and its material's modulus of elasticity E . Use the basic method of dimensional analysis to show that:

$$CEd = F_{CEd} \left(\frac{l}{d} \right).$$

- 2.23.** Experiments were conducted to determine the specific form of the function $F_{CEd}(l/d)$ found in Problem 2.22. In these experiments it was found that a plot of $\log_{10}(CEd)$ against $\log_{10}(l/d)$ has a slope of 3 and an intercept on the $\log_{10}(CEd)$ scale of -0.60 . Show that the deflection under a load P can be given in terms of the second moment of area I of the cross section ($I = d^4/12$) as:

$$\text{deflection} = \text{load} \times \text{compliance} = P \times C = \frac{Pl^3}{48EI}.$$