6
Traffic Flow Models

People like to drive, especially in the United States. In fact, we can often tell where people come from by how they refer to highways: people on America's east coast talk about taking *the turnpike* (or *the 'pike*) or *the interstate*, while on the west coast we get on *the freeway* or we take *the 5* or *the 101*, referring to a particular highway by its number. In order to design the roads and the cars that enable and facilitate such personal transportation, we model both the behavior of *individual* cars with their drivers in a (single) line of autos, and that of *groups* of cars in one or more lanes of traffic. However, our concern is not with modeling the ergonomics of operating a car. Rather, we focus on the interactions of autos on single highway lanes, both individually and in dense lines.

6.1 Can We Really Make Sense of Freeway Traffic?

No matter how we refer to traffic arteries, the flow of traffic on them is modeled, analyzed, and predicted with *traffic flow theory*, which we now detail at two levels. The *macroscopic modeling* of traffic assumes a sufficiently large number of cars in a lane or on a road such that each stream of autos can be treated as we would treat fluid flowing in a tube or stream. Thus, to maintain the biological metaphor, traffic flow is treated as a flow of a fluid *field* in an artery. Macroscopic models are expressed in terms of three gross or *average* variables for a whole line of traffic: the number of cars passing a fixed point per unit of time, called the *rate of flow*; the distance covered per unit time, the *speed of the traffic flow*; and the number of cars in a traffic line or column of given length, which we identify as the *traffic density*. The relationship between the speed and the density is embodied by macroscopic modelers in a plot of these two variables called the *fundamental diagram*. We also invoke the *continuum hypothesis* (viz. Section 4.7.2) to confirm that it is appropriate to (mathematically) treat the traffic as a field.

The second level of traffic modeling, *microscopic modeling*, addresses the interaction of individual cars in a line of traffic. Microscopic models describe how an individual *follower* car *responds* to an individual *leader* car by modeling its acceleration as a function of various perceived *stimuli*, which might be the distance between the leader and follower cars, the relative speeds of the two cars, or the reaction time of the operator of the follower car. Car-following models come in several varieties, and they can be used to construct the speed-density curves that are the underpinning of macroscopic modeling. Such speed-density plots, supported by data taken from real traffic arteries, enable traffic experts to model and understand road or freeway capacity as a function of traffic speed and density—even if everyday drivers feel they do not fully "understand" what is happening around them. (The microscopic models are also used to support the modeling of vehicular *control*, that is, to implement control strategies that enable lines of traffic to maintain high flow rates at high speeds. However, we will not delve into control theory and its applications here.)

We will start our brief overview of traffic modeling at the macroscopic level, applying conservation principles for cars aggregated into a field (or sufficiently large collection of cars) to define the fundamental diagram for the flow of traffic on a highway populated with multiple vehicles. Then we will examine how the continuum hypothesis influences our view of individual cars (and drivers), as a guide to developing car-follower models that model the interaction between a single car as its driver reacts to another auto immediately ahead. These car-follower models are then used to derive the speed-density relationships that allow us to put specific models and numbers into the more general macroscopic traffic flow theory.

6.2 Macroscopic Traffic Flow Models

Why? We start by asserting the validity of an analogy, namely, that the flow of a stream of cars can be modeled as a field, much as we would model the flow of a fluid. Thus, the collection of cars taking the 10 east out of Los Angeles on any given evening is mathematically similar to the flow of blood in an artery or water in a home piping system. We want to relate the speed of a line of traffic to the amount of traffic in that line (or lane). We use three variables to describe such traffic flows: **Find?**

- the rate of flow, $q(x, t)$, measured in the number of cars per unit time;
- the density of the flow, $\rho(x, t)$, which is the number of vehicles per unit length of road; and
- the speed of the flow, $v(x, t)$.

How are these three variables related?

6.2.1 Conservation of Cars

How? We can provide one answer to the foregoing question by applying the conservation principle embodied in eqs. (1.1) and (1.2) to traffic moving (in one direction) along an arbitrary stretch of a road. The conservation principle states that the change in the number of cars within that stretch of road results from the flow of traffic into and out of that road interval, and from the generation or consumption of cars within the interval. **Assume?** Notwithstanding the occasional pictures we have all seen of horrific mega-accidents that occur during severe fogs or major storms, we will (safely) assume that cars are neither generated nor consumed within that road interval.

Thus, imagine a coordinate, x, along a particular stretch or interval of road under consideration that has endpoints defined by $x = x$ and $x = x + \Delta x$. The number of cars within this road interval of length Δx is given by $\Delta N(x, t)$. Given our assumption that we will neither generate or consume cars, the conservation principle of eq. (1.2) states that the change in the number of cars within the interval $\Delta N(x, t)$ during a time interval Δt is, in the limit, equal to the *rate of traffic flow*, $q(x, t)$:

$$q(x, t) \equiv \lim_{\Delta t \to 0} \frac{\Delta N(x, t)}{\Delta t}. \tag{6.1}$$

The change in the number of cars within the road interval, $\Delta N(x, t)$, is simply the difference between the number of cars going in and out of that stretch of road at each end, $N(x, t)$ and $N(x + \Delta x, t)$, respectively:

$$\Delta N(x, t) = N(x, t) - N(x + \Delta x, t), \tag{6.2}$$

If Δx denotes the length of road interval that is traveled during the time, Δt, the statement of conservation of cars (6.1) can also be written as

$$q(x, t) = \lim_{\Delta t \to 0} \frac{\Delta N(x, t)}{\Delta x} \left(\frac{\Delta x}{\Delta t} \right), \tag{6.3}$$

where the fraction introduced in eq. (6.3) is the speed of the traffic, $v(x, t)$, in the interval:

$$v(x, t) = \left(\frac{\Delta x}{\Delta t}\right). \tag{6.4}$$

Equations (6.2) and (6.4) are now substituted into the conservation of cars (6.3) to yield

$$q(x, t) = \left(\lim_{\Delta x \to 0} \frac{N(x, t) - N(x + \Delta x, t)}{\Delta x}\right) v(x, t). \tag{6.5}$$

Note that the limit in eq. (6.5) is now taken as $\Delta x \to 0$, and that its dimensions correspond to the number of vehicles per unit length of road, which we define as the *density of the traffic flow*:

$$\rho(x, t) \equiv \lim_{\Delta x \to 0} \frac{N(x, t) - N(x + \Delta x, t)}{\Delta x}. \tag{6.6}$$

Thus, eq. (6.5) can be rewritten for the last time to cast the *principle of conservation of cars* in the form

$$q(x, t) = \rho(x, t)\, v(x, t). \tag{6.7}$$

Beyond preserving the notion that "what goes in must go out," what does eq. (6.7) mean? First, we note that the equation is dimensionally consistent and correct (see Problem 6.1). Second, we note that eq. (6.7) can be shown to make "physical" sense by a rather simple argument derived by looking at two different ways of counting the number of cars passing a (specified) point on the road during a very small time interval.

One measure of the traffic count is that the number of cars, ΔN, passing a point during a time interval, Δt, is simply the product of the flow rate, q, and the time interval: $\Delta N = q\Delta t$. The second measure count assumes that during the same small interval of time a car moving with a speed, v, will cover a distance, $\Delta x = v\Delta t$. The number of vehicles passing through that distance is found from another simple product: of density, ρ, times distance: $\Delta N = \rho\Delta x$. Hence, equating the two measures of the number of cars passing a point yields the result

$$q\Delta t = \rho\Delta x, \tag{6.8}$$

which is clearly an averaged version of eq. (6.7) that accords well with this elementary physical reasoning (see Problem 6.2).

We also observe that the single equation (6.7) is expressed in three variables: q, ρ, and v. Therefore, it is of very limited use in this form without substantial further information. However, it is clear that traffic density, ρ, and speed, v, are the *two fundamental traffic variables* because we can determine the rate, q, at which traffic flows by inserting them into eq. (6.7).

Further, if we could relate speed directly to density, i.e., $v = v(\rho)$, then we could write a direct relationship between the traffic flow rate, q, and the density, ρ:

$$q(\rho) = \rho\, v(\rho). \tag{6.9}$$

As we will see in Section 6.2.3, plots of traffic flow rate, q, against density, ρ, are so widely used in modeling traffic flow that they are identified under the rubric of the *fundamental diagram of road traffic.*

Speed-density relationships (e.g., $v = v(\rho)$) are clearly central to our understanding of traffic flow, so we turn to them next.

Problem 6.1. Confirm that eq. (6.7) is dimensionally correct.

Problem 6.2. Explain which variables were averaged, and how, over the intervals of distance (Δx) and time (Δt) in the heuristic derivation of eq. (6.8)?

6.2.2 Relating Traffic Speed to Traffic Density

Even inexperienced drivers would agree that traffic speed and traffic density are related. Drivers speed up when traffic is sparse, and they slow down (perhaps involuntarily!) to clog up arteries when traffic is thick. Thus, we are tempted to postulate that there is a direct relationship between traffic speed and traffic density: **Assume?**

$$v = v(\rho). \tag{6.10}$$

Let us now reason a bit further about this relationship to determine any conditions that need to be applied to any particular functional form, $v(\rho)$, that might be proposed.

Building on the intuition just mentioned, we expect that a driver will drive fastest, $v_{\max}$, when the density is at its smallest value, $\rho \to 0$. The speed decreases as the density increases, which is a statement about the slope of the v versus ρ curve. Finally, traffic grinds to a halt, $v = 0$, at some maximum or *jam* density, ρ_{jam}, presumably when the traffic is bumper-to-bumper. We can summarize these experience-born intuitions in mathematical requirements on the function, $v(\rho)$: **Assume?** **Assume?**

$$v(\rho = 0) = v_{\max}, \tag{6.11a}$$

$$\frac{dv}{d\rho} \leq 0, \tag{6.11b}$$

$$v(\rho = \rho_{\text{jam}}) = 0. \tag{6.11c}$$

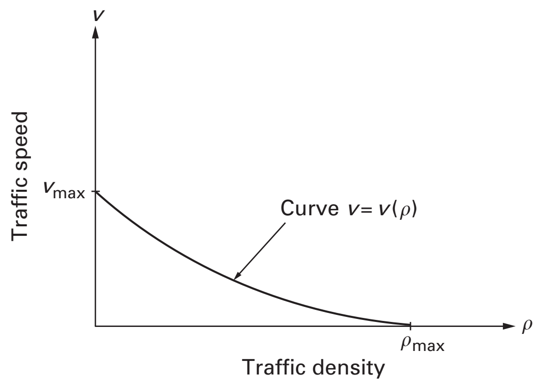


Figure 6.1 A generic schematic of the variation of traffic velocity with density. It displays the endpoints, [$(0, v_{max})$ and $(\rho_{max}, 0)$, respectively], and shows that the slope is always non-positive, $dv/d\rho < 0$, which results from our experience that traffic speed drops off as traffic density increases.

We can also display these results graphically, in the generic curve shown in Figure 6.1. Note that the precise shape of the curve is unknown; only the endpoint values and the sign of the slope are specified at this point.

The elementary modeling assumptions just outlined do not exhaust all of the possibilities, although experience suggests that eqs. (6.10) and (6.11) adequately reflect the behavior of traffic that is accelerating or decelerating. Models behind traffic speed-density relations will reflect human behavior—rather than mechanical laws—because they reflect how drivers respond to stimuli. That is, drivers can respond to perceived distances between cars, to relative speeds, to the perceived density further down the road, and so on. In fact, speed-density relations such as eq. (6.10) are found both from empirical data and from the very stuff of the modeling of car-following interactions that we address in Section 6.3.

6.2.3 Relating Traffic Flow to Traffic Density: The Fundamental Diagram

Why? From the viewpoint of the traffic engineer who is designing a road and all of its facilities (including entrance and exit ramps, traffic signs and signals, toll booths, etc.), the most relevant variable is the *capacity* (or maximum flow rate) that the road system must accommodate, as reflected in its traffic

Given? flow rate, $q(x, t)$. For macroscopic models we can take the speed to be *homogeneous*, which means that it does not explicitly depend on the road coordinate, x, or on time, t. Then, we can write $v = v(\rho)$, anticipating as in eq. (6.9), that traffic flow ultimately depends only on the density, ρ.

We can now extend our qualitative analysis of the speed-density relationship (of Section 6.2.2) to the relationship between the traffic flow rate and the density. Thus, because a driver's fastest speed, v_{max}, occurs when the density is at its smallest, $\rho = 0$, eq. (6.9) tells us that $q(\rho = 0) = 0$, that is, that the flow rate is zero. Similarly, when traffic slows to a halt at its maximum density, $v(\rho_{jam}) = 0$, eq. (6.9) tells us once again that the traffic flow rate is zero: $q(\rho_{jam}) = \rho_{jam} v(\rho_{jam}) = 0$. The traffic flow rate must be positive for all values of the density ($0 < \rho < \rho_{jam}$), and must attain its maximum value q_{max} somewhere in that interval. Further, the slope of the traffic flow rate is given by (see Problem 6.3):

$$\frac{dq}{d\rho} = v(\rho) + \rho \frac{dv}{d\rho}. \tag{6.12}$$

The qualitative results just found are embodied in the generic curve shown in Figure 6.2, which is called the *fundamental diagram of traffic flow*. As with Figure 6.1, the precise shape of the curve is unknown: the endpoint values are specified and the variation of the slope can be inferred (see Problem 6.4).

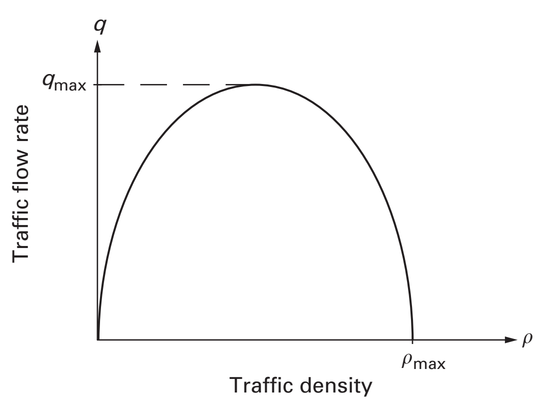


Figure 6.2 A generic schematic of the variation of the traffic flow rate with density. It displays the endpoints, [(0, 0) and (ρ_{max}, 0), respectively], and shows that the slope is positive until the maximum flow rate or *capacity*, q_{max}, is reached, and negative thereafter.

To make some of these qualitative ideas more specific, consider the following linear speed-density relationship:

$$v(\rho) = v_{\text{max}}\left(1 - \frac{\rho}{\rho_{\text{jam}}}\right). \tag{6.13}$$

This relationship clearly satisfies (see Problem 6.5) all of the conditions required by eqs. (6.11a–c). Moreover, as the simplest (linear) mathematical expression that satisfies these conditions, it is particularly attractive as a "building block" for further modeling, provided that it adequately models reality. When substituted into eq. (6.9), it produces a relationship for the traffic flow rate as a function of density that is *parabolic*:

$$q(\rho) = v_{\text{max}}\left(\rho - \frac{\rho^2}{\rho_{\text{jam}}}\right). \tag{6.14}$$

The maximum flow rate occurs when its slope vanishes:

$$\frac{dq(\rho)}{d\rho} = v_{\text{max}}\left(1 - \frac{2\rho}{\rho_{\text{jam}}}\right) = 0. \tag{6.15}$$

Equation (6.15) shows that the maximum traffic flow rate under these assumptions occurs at the mid-point of the fundamental diagram, when $\rho = \rho_{\text{jam}}/2$, and that its value is

$$q_{\text{max}} = \frac{1}{4}\rho_{\text{jam}} v_{\text{max}}. \tag{6.16}$$

So, is the linear speed-density relationship of eq. (6.13) just a nice demonstration model, or does it have any real utility or validity in modeling traffic flow? As a matter of fact, it is useful. In studies conducted for the Lincoln, Holland, and Queens-Midtown Tunnels leading into New York's Manhattan island, for example, the linear speed-density relationship has been shown to be a very good approximation to the central (and dominant) part of the speed-density data gathered empirically. Such a curve is shown in Figure 6.3. We will return to this point in Section 6.3 because car-following models are expressly used to derive speed-density relationships.

Problem 6.3. Demonstrate that eq. (6.12) is correct.

Problem 6.4. Confirm qualitatively that eq. (6.12) produces the shape of the fundamental diagram of road traffic shown in Figure 6.2.

Problem 6.5. Show that the relationship (6.13) satisfies the conditions defined in eqs. (6.11a–c).

Problem 6.6. Derive and sketch the fundamental diagram for the speed-density relationship

$$v(\rho) = v_{\max}\left(1 - \left(\frac{\rho}{\rho_{\text{jam}}}\right)^2\right).$$

Problem 6.7. Derive and sketch the fundamental diagram for the speed-density relationship

$$v(\rho) = v_{\max}\left(1 - \left(\frac{\rho}{\rho_{\text{jam}}}\right)^m\right).$$

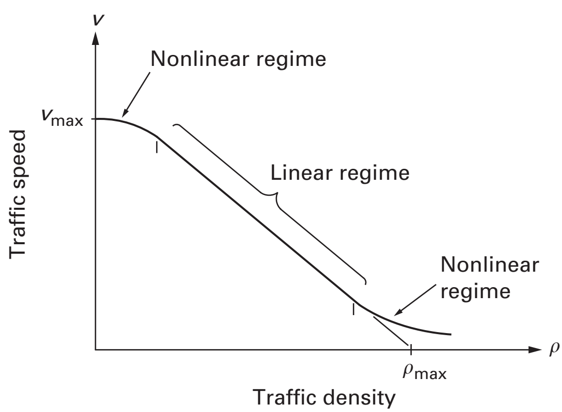


Figure 6.3 Another generic view of the variation of traffic velocity with density, based on the results often obtained when data is gathered for particular traffic systems. In addition to displaying the endpoints, $[(0, v_{\max}), (\rho_{\max}, 0)]$, and the non-positive ($dv/d\rho < 0$) slope behavior, it shows that a significant portion of the curve can be modeled by a linear speed-density relationship.

6.2.4 The Continuum Hypothesis in Macroscopic Traffic Modeling

The macroscopic traffic flow analysis we have done so far has been predicated on the proposition that we could treat a line of traffic in the same way that we would model the flow of a fluid through an artery or tube, that is, as a field. This means that the traffic line contains enough cars that instead of worrying about the speed of the ith car, $v_i(x, t)$, we choose to deal with a speed *field* in which every point along the x axis is assigned a unique speed $v(x, t)$. Thus, we have replaced the line of discrete cars at coordinates, $x = x_i$, by an infinite sequence of points, each having a unique speed expressed by the continuous function, $v(x, t)$. This is an application of the *continuum hypothesis* that we discussed in Section 4.7.2. Taking advantage of the continuum hypothesis allows us to deal with continuous fields (e.g., smooth curves) instead of discrete elements (e.g., histograms), which often makes the mathematics of model building much nicer. However, it carries drawbacks: in the present model, for example, we could not include cars overtaking and passing each other because that would require some points on the x axis to have two different speeds!

How many cars do we need for a macroscopic analysis? The answer depends on how we characterize the number of cars. We saw in Section 6.2.1 that we could measure the number of cars in two ways. One way is to stand at a fixed point and count the number of cars passing by during a fixed time interval, thus finding the traffic flow rate, $q(x, t)$, with units of cars per unit of time. The second way requires counting the number of cars in a given length of road and so determining the traffic density, $\rho(x, t)$, with units of cars per unit of distance. (As a practical matter, the density would be determined from aerial photographs of a given length of road.) In both instances we must ask whether our counting intervals are sufficiently long, that is, have we taken enough time to measure the traffic flow or enough distance to measure the density?

To measure the density, we must choose a length of road that is (1) not so short that we too often see fractions of cars or intervals with no cars at all, and (2) not so long that the meaningful fluctuations would simply cancel out. For example, a spatial count over the length of Interstate 5 between Los Angeles and San Francisco—about 350 miles—would miss both the buildup at cities along the way and the long stretches through farm country with sparse amounts of traffic. Figure 6.4 shows a conceptual sketch for just such a measurement, showing the variation of traffic density with the length of the measurement interval. (Note how similar it is to its cousin in Figure 4.9!) It illustrates the discontinuities arising from the fluctuations when the measuring interval is too short, and it shows the decline in the density when the measuring length becomes so long that the meaningful variations disappear. The central portion shows a regime where the *local* density is relatively constant. It is for this region that we can model our traffic density with a continuous field $\rho(x, t)$, in much the same way we replaced the speeds of individual cars with the continuous speed field, $v(x, t)$.

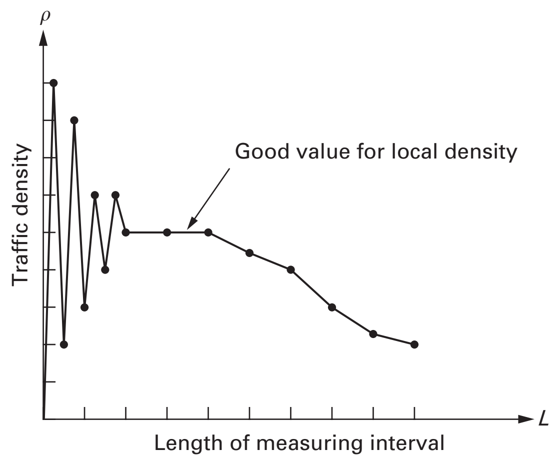
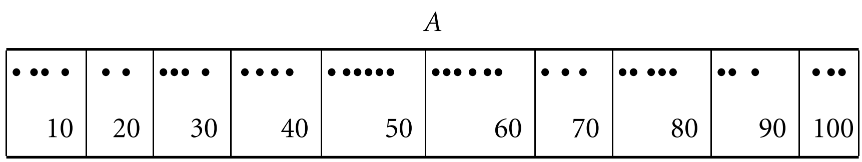


Figure 6.4 A conceptual plot of the variation of traffic density, ρ, against the length of the measuring interval. It shows that the central portion of the curve defines a useful approximation of the *local* traffic density that is (1) preceded by a regime where the density fluctuates too much because the measuring interval is too short, and (2) followed by a regime where the density progressively falls off because the measuring interval is too long.

A comparable situation obtains if the traffic flow rate, $q(x, t)$, is the measurement of choice. Here it is the length of the time interval that must be "just right." Short intervals before and after the change of a traffic light, say from red to green, would show no cars before and a sudden burst after. Similarly, counting by days would almost certainly cover up the peaks generated by morning and evening rush hours. Thus, again, there is a balancing act that must be performed in order to get the time measurement interval properly set.

To sum up, the continuum hypothesis enables us to deal with *averaged* or gross variables of traffic speed, density, and flow rate that do *not* pertain to individual cars or vehicles, but to the fields that represent them. And, these fields are good models, or good representations of reality, if we have done our *scaling* properly in choosing the proper measurement intervals, that is, if we have properly set the measurement scales. **How?**

Problem 6.8. Consider a road segment 0.5 mi long that is divided into 10 equally-spaced intervals. There are 40 cars on the road, spaced as shown below, where the density of the dots represents the traffic density. Find a "good" value for the local density at point A in terms of the number of cars per mile, assuming for simplicity that each car has zero length.



6.3 Microscopic Traffic Models

We now turn from macroscopic models that use averaged variables to Why? *microscopic* models that look at individual cars. Our interest is in using the microscopic models to develop the traffic speed-density relations that we need to do macroscopic evaluations of capacity, which we require if Find? we're going to design highway systems. As we noted in Section 6.2.2, we are looking for models that describe how drivers *respond* to the *stimuli* of their traffic situations. The driver will *perceive* a variety of stimuli, including the distance between vehicles, their relative speed, and their perceived relative acceleration. We thus seek psychological, not mechanical, models in order to model human behavior. The driver's response will depend on the responder's *sensitivity* to the given stimuli, as well as on the speed with which the response is undertaken. Thus, some time delay should also be incorporated into such models.

6.3.1 An Elementary, Linear Car-following Model

Imagine a line of cars traversing a given road, as shown in Figure 6.5. Each car is identified by a discrete coordinate that varies in time, so that the Assume? location of the nth car is given by $x_n(t)$. We also assume that the line has a reasonable value of local density and does not permit passing or overtaking. How? Then the basic "equation" of car-following for such a single lane of traffic is the psychological one:

$$\text{response} = \text{sensitivity} \bullet \text{stimulus}. \tag{6.17}$$

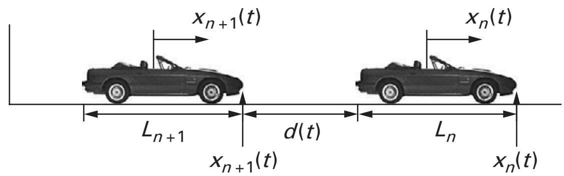


Figure 6.5 The nomenclature for a line (or lane) of cars on a highway of total length, L_R. Each car has the same length, L, and is separated from its neighbors by a common distance, $d(t)$. The discrete functions, $x_{n+1}(t)$ and $x_n(t)$, represent, respectively, the coordinates of the *follower* and *leader* cars.

The response will generally be modeled as the acceleration of the $(n+1)$st *follower* car, $\ddot{x}_{n+1}(t)$, as it moves behind the nth *leader* car. The stimulus will be modeled in terms of the coordinate of the follower car *relative to the leader car*, which can in turn be written in terms of the traffic density, ρ. The acceleration is then integrated to determine the speed of that car as a function of the traffic density, which is the input we require for our macroscopic modeling. **Predict?**

Consider a simple linear car-following model in which the driver of the follower car responds to the speed of the leader car relative to the follower car:

$$\frac{d^2x_{n+1}(t)}{dt^2} = -K_p\left(\frac{dx_{n+1}(t)}{dt} - \frac{dx_n(t)}{dt}\right). \tag{6.18}$$

The coefficient, K_p, introduced here is a sensitivity parameter that has dimensions of per unit time. Note, that with $K_p > 0$, the follower car will decelerate to avoid hitting the car in front if it is slowing down, relatively speaking. We will discuss this in further detail later.

We can model the time it takes the following driver to respond to events by building in a *reaction* time that slows the follower's acceleration by the *delay* time T:

$$\frac{d^2x_{n+1}(t+T)}{dt^2} = -K_p\left(\frac{dx_{n+1}(t)}{dt} - \frac{dx_n(t)}{dt}\right). \tag{6.19}$$

Assuming that the sensitivity parameter, K_p, is a constant, eq. (6.19) is a linear ordinary differential equation with constant coefficients that can be integrated once to yield

$$\frac{dx_{n+1}(t+T)}{dt} = -K_p(x_{n+1}(t) - x_n(t)) + C_{n+1}, \tag{6.20}$$

where C_{n+1} is the arbitrary constant, with dimensions of speed, that results from the integration just performed. Note that eq. (6.20) clearly relates the speed of the follower car to the distance maintained between the follower and leader cars. Thus, it is a natural precursor of the speed-density relationship that we seek.

Assume? Let us further assume that all of the cars have the same length, L, and that the spacing between common points on any pair of cars (see Figure 6.5) is given by $d(t)$:

$$d(t) = x_n(t) - L - x_{n+1}(t). \tag{6.21}$$

It then follows that the number of cars, N_R, found in a stretch of road of length, L_R, is

$$N_R = \frac{L_R}{L + d(t)}, \tag{6.22}$$

which means that the density of cars on that road is

$$\rho = \frac{L_R}{N_R} = \frac{1}{L + d(t)} = \frac{1}{x_n(t) - x_{n+1}(t)}, \tag{6.23}$$

where we have used the spacing defined in eq. (6.21) to obtain the final form of eq. (6.23). Thus, we have in eq. (6.23) a relationship between the (macroscopic) traffic density, ρ, and the (microscopic) coordinates of the leader and follower cars.

There is an important point about the units of eq. (6.23) that should be kept in mind. With particular reference to the units still used by American traffic engineers, both car lengths and inter-vehicle distances are typically measured in feet, while density is expressed in units of vehicles per mile. Thus, for numerical calculations, eq. (6.23) should be written in consistent numerical units:

$$\rho = \frac{5280}{L + d(t)} \left(\frac{\text{vehicles}}{\text{mile}}\right). \tag{6.24}$$

Assume? Let us still further assume, for now at least, that the traffic flow is in a *steady state*, by which we mean that all of the cars are traveling at the same speed. Then

$$\frac{dx_{n+1}(t+T)}{dt} = \frac{dx_{n+1}(t)}{dt} \equiv v. \tag{6.25}$$

Equation (6.24) shows a relationship between the (macroscopic) speed, v, and the (microscopic) speeds of any of the follower cars. Additionally, for this steady state, the arbitrary constant C_{n+1} is the same for any adjacent pair of cars. Thus, we can now substitute eqs. (6.23) and (6.25) into eq. (6.20) to find

$$v = \frac{K_p}{\rho} + C. \tag{6.26}$$

The constant, C, can be determined from the condition cited in eq. (6.11c), namely, that the speed is zero when the density is at its maximum or jam value. Hence it follows that

$$v = K_p \left(\frac{1}{\rho} - \frac{1}{\rho_{\text{jam}}} \right). \tag{6.27}$$

The speed-density relationship of eq. (6.27) is sketched in Figure 6.6.

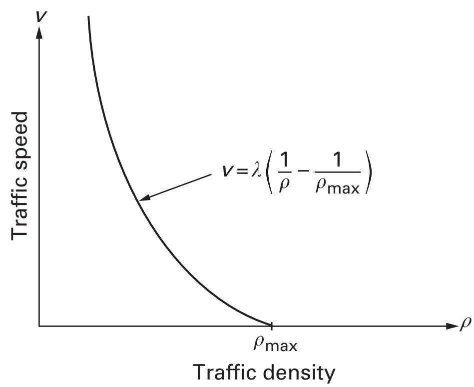


Figure 6.6 A schematic curve illustrating the traffic speed-density relationship [see eq. (6.27)] corresponding to a linear car-following model in which the driver responds to the relative speed of the car ahead.

The curve shown in Figure 6.6 seems reasonable enough (see Problem 6.10), except for the fact that it shows an infinite speed as the density goes to zero, a result that hardly seems credible. This is an almost classical modeling dilemma: we have a model that seems reasonable and credible over a good portion of the relevant domain, but that crashes in some region. Can this model be improved or fixed? It can be fixed, or improved; it depends on what we want from this model. **Valid?** **Improve?**

Fixing the high (infinite at $\rho = 0$) speed at small values of the density is straightforward enough. All we need do is stipulate that a maximum speed applies for all values of density below some (specified) critical density. This seems like a reasonable fix that roughly accords with our everyday driving experience. This fix is shown in Figure 6.7 and in eqs. (6.28a–b):

$$v(\rho) = \begin{cases} v_{\text{max}} & \rho < \rho_{\text{crit}} \\ K_p \left(\frac{1}{\rho} - \frac{1}{\rho_{\text{jam}}} \right) & \rho \geq \rho_{\text{crit}} \end{cases} \tag{6.28a}$$

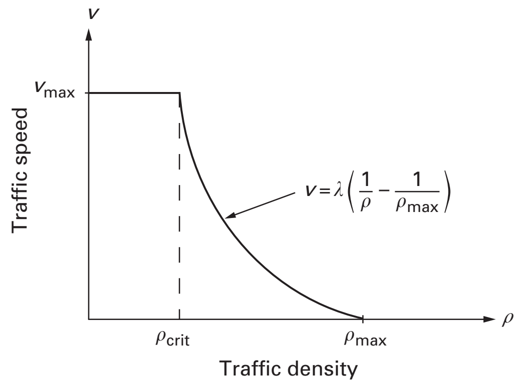


Figure 6.7 A schematic curve illustrating the traffic speed-density relationship [eqs. (6.28a–b)] corresponding to the *fixed* linear car-following model in which the driver responds to the relative speed of the car ahead—except at small values of the density, $\rho < \rho_{crit}$, for which the maximum speed has a fixed upper limit of $v = v_{max}$.

and

$$\rho_{crit} = \left(\frac{v_{max}}{K_p} + \frac{1}{\rho_{jam}}\right)^{-1}. \tag{6.28b}$$

The traffic flow rate corresponding to this fixed speed-density relationship is found as:

$$q(\rho) = \begin{cases} \rho v_{max} & \rho < \rho_{crit} \\ K_p\left(1 - \dfrac{\rho}{\rho_{jam}}\right) & \rho \geq \rho_{crit} \end{cases} \tag{6.29}$$

The traffic flow rate, pictured in Figure 6.8, increases linearly with density (from zero), and reaches its maximum value, the capacity, when $\rho = \rho_{crit}$:

$$q_{max} = q(\rho_{crit}) = \rho_{crit} v_{max} = K_p\left(1 - \frac{\rho_{crit}}{\rho_{jam}}\right). \tag{6.30}$$

For density values $\rho \geq \rho_{crit}$, the traffic flow rate decreases linearly with ρ from its maximum value at $\rho = \rho_{crit}$ until it vanishes altogether at $\rho = \rho_{jam}$.

Verified? How good is this model? As luck would perhaps have it, having just fixed a model that is incredible (literally!), we are still left with one that does compare well with some available data. In Figures 6.9 and 6.10 we show measurement data made in Orange County, California, on the I–405 freeway. It yields reasonable values of the jam (or maximum) density and, as shown in Figure 6.10, the shape of the resulting traffic flow rate curve

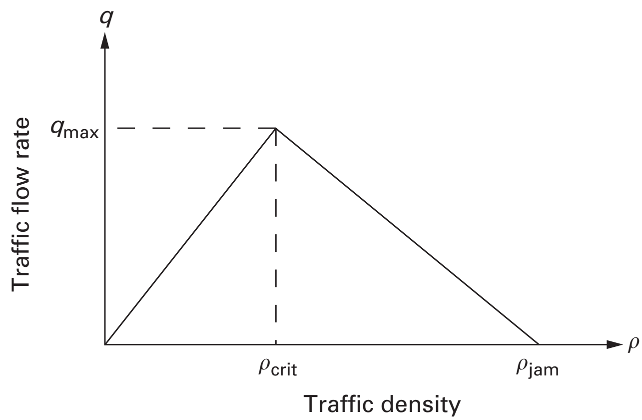


Figure 6.8 A schematic curve illustrating the relationship between the traffic flow rate and the density [eq. (6.29)] for the *fixed* linear car-following model in which the driver responds to the relative speed of the car ahead. Note that the maximum traffic flow rate $q = q_{max}$ occurs when $\rho = \rho_{crit}$.

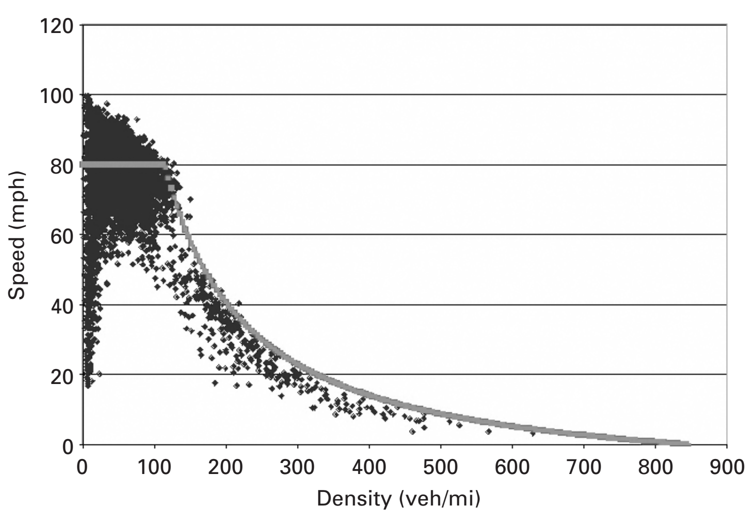


Figure 6.9 Some traffic speed-density data measured for the I–405 freeway in Orange County, California, plotted along with corresponding results from the *piecewise linear* or *triangular* car-following model [eq. (6.38)] (Recker, 2003). The corresponding parameter values are $S_f = 80$ mph, $q_{crit} = 2300$ cars/hr, and $\rho_{jam} = 211$ cars/mi.

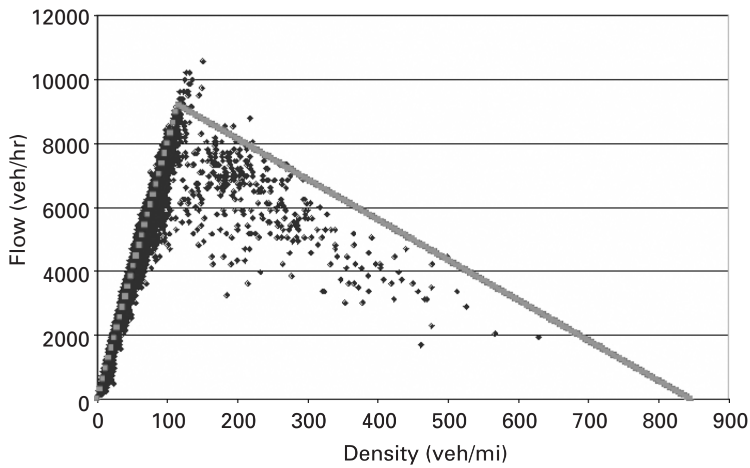


Figure 6.10 Some traffic flow rate data measured for the I–405 freeway in Orange County, California, plotted along with the *piecewise linear* or *triangular* car-following model [eq. (6.35)] (Recker, 2003). The corresponding parameter values are $S_f = 80$ mph, $q_{\text{crit}} = 2300$ cars/hr, and $\rho_{\text{jam}} = 211$ cars/mi.

follows the data for traffic parameter values that are not uncommon on California freeways, including speeds up to 80 mph and jam densities of 211 veh/mi/lane that correspond to vehicles stopped at 25 ft separation.

Another aspect of this model is worth noting. One of the heuristics or rules of thumb offered by state Departments of Motor Vehicles (DMV) is that drivers should maintain a distance behind the car immediately in front that is equal to one car length, L (ft.), for each increment of 10 mph of the car's speed. Thus, the DMV heuristic would require that

$$d(t) = \left(\frac{L}{10}\right) v. \tag{6.31}$$

If eq. (6.31) is substituted into our previous, units-corrected definition of the traffic density (6.24), we immediately obtain a speed-density relationship

$$\rho = \frac{5280}{L + (L/10)v},$$

that can be recast in the form:

$$v = \frac{5280(10)}{L}\left(\frac{1}{\rho}\right) - 10. \tag{6.32}$$

Equation (6.32) bears an unmistakable resemblance to the result (6.27) derived just above (see Problems 6.13–6.15).

Problem 6.9.	Derive eq. (6.21) from Figure 6.5.
Problem 6.10.	Determine whether or not eq. (6.26) satisfies each of the three conditions in eqs. (6.11a–c).
Problem 6.11.	Derive the result presented in eq. (6.28b). Is it dimensionally correct?
Problem 6.12.	Confirm the traffic flow rate results shown in eqs. (6.29).
Problem 6.13.	Determine the values of the constants, K_p and ρ_{jam}, that make eqs. (6.27) and (6.32) identical.
Problem 6.14.	Why does the DMV model produce the same form (and numbers) as the speed-sensitive car-following model?
Problem 6.15.	What is the physical interpretation of ρ_{jam} for the DMV model?

6.3.2 An Alternate Derivation of the Same Model

Suppose we want to derive the above model using an empirical, yet "mechanical" approach. We know that flow rate increases with density until it reaches a critical value, and then it decreases to zero at the jam density. Thus—without benefit of the car-following model (6.25) or the data we have already seen in Figure 6.10!—we assume a priori that the traffic flow rate will behave in a piecewise linear fashion, in the following *triangular* traffic flow rate:

Improve?

Assume?

$$q(\rho) = \begin{cases} A\rho & \rho < \rho_{\text{crit}} \\ B\left(1 - \dfrac{\rho - \rho_{\text{crit}}}{\rho_{\text{jam}} - \rho_{\text{crit}}}\right) & \rho \geq \rho_{\text{crit}} \end{cases} \tag{6.33}$$

where the constants A and B are determined from the requirement that $q(\rho)$ be continuous at $\rho = \rho_{\text{crit}}$, that is,

$$q(\rho = \rho_{\text{crit}}) = q_{\text{crit}}. \tag{6.34}$$

Thus, eq. (6.33) becomes

$$q(\rho) = \begin{cases} q_{\text{crit}}\left(\dfrac{\rho}{\rho_{\text{crit}}}\right) & \rho < \rho_{\text{crit}} \\ q_{\text{crit}}\left(1 - \dfrac{\rho - \rho_{\text{crit}}}{\rho_{\text{jam}} - \rho_{\text{crit}}}\right) & \rho \geq \rho_{\text{crit}} \end{cases} \tag{6.35}$$

The speed-density relationship corresponding to the traffic flow rate (6.35) is then found by applying the relationship (6.9) between the traffic flow rate and the speed, so that

$$v(\rho) = \begin{cases} \dfrac{q_{\text{crit}}}{\rho_{\text{crit}}} & \rho < \rho_{\text{crit}} \\ \dfrac{q_{\text{crit}}}{\rho_{\text{crit}}} \left(\dfrac{\dfrac{\rho_{\text{jam}}}{\rho} - 1}{\dfrac{\rho_{\text{jam}}}{\rho_{\text{crit}}} - 1} \right) & \rho \geq \rho_{\text{crit}} \end{cases} \tag{6.36}$$

While the speed-density relationship in eq. (6.36) does not have the nice, linear properties of the speed-density of eq. (6.13), we have maintained the corresponding piecewise linear flow-density relationship. Equations (6.35) and (6.36) have the same form as, respectively, eqs. (6.29) and (6.28), although they were derived by very different means!

One interesting version of the results in eq. (6.36) is their presentation in terms of a parameter called the *free-flow speed*, S_f, which is the speed at which a driver would travel if all alone on the road, that is, if the density were zero. From the first of eq. (6.36) we find that

$$S_f = \frac{q_{\text{crit}}}{\rho_{\text{crit}}}, \tag{6.37}$$

from which it follows that eqs. (6.36) now become:

$$v(\rho) = S_f \begin{cases} 1 & \rho < \rho_{\text{crit}} \\ \left(\dfrac{\dfrac{\rho_{\text{jam}}}{\rho} - 1}{\dfrac{\rho_{\text{jam}}}{\rho_{\text{crit}}} - 1} \right) & \rho \geq \rho_{\text{crit}} \end{cases} \tag{6.38}$$

Verified? Equations (6.38) and (6.35), with parameter values of $S_f = 80$ mph, $q_{\text{crit}} = 2300$ cars/hr, and $\rho_{\text{jam}} = 211$ cars/mi, are shown in Figures 6.9 and 6.10, together with data taken from the I–405 freeway measurements. We see that the agreement is quite good over most of the range of density for both the speed and the traffic flow.

6.3.3 Comments on Car-following Models

It is worth noting that the two models just presented were found in very different ways. The elementary and *fixed* car-following models of Section 6.3.1 were derived from a stimulus-response model that was re-worked into a speed-density relationship, from which we then obtained the traffic flow rate. The revised model presented in Section 6.3.2 was found by starting with traffic flow rate data and trying to create a model to match that data. Indeed, we have not gone so far as to find a matching stimulus model for the improved model. Does that matter?

The answer is a familiar one: it depends. If our principal goal is the one we claimed earlier, that of modeling capacity, then it matters less which of the two approaches we use as long as we can validate and verify the results. On the other hand, in an emerging area of transportation engineering, efforts are being made to model the *control* of vehicles, with the aim of trying to maximize the flow of traffic by more effectively controlling how each vehicle is driven. This area encompasses a number of exciting prospects that are, unfortunately, beyond our present scope. Achieving results in the latter case means that stimulus-response control modeling will be required, while "only" good modeling of traffic speed and traffic flow rate is required for capacity-based engineering to move forward.

6.4 Summary

This chapter has introduced some of the most fundamental ideas of traffic modeling as they are applied in the engineering of traffic systems. We described macroscopic models that predict the average variables of traffic density and traffic flow rates because they are very important for calculating the *capacity* of roads and highways. We then pointed out the role of scaling and of the continuum hypothesis in moving from macroscopic models to microscopic and in beneficially integrating the two. We introduced microscopic models that predict how speed varies with driver sensitivities and responses to various traffic stimuli because they provide a basis for obtaining the gross traffic density and flow rates needed in macroscopic models. Finally, we also noted in passing that the microscopic models are increasingly used to investigate the control of individual vehicles, as well as lines (or lanes) of vehicles.

6.5 References

R. E. Chandler, R. Herman, and E. W. Montroll, "Traffic Dynamics: Studies in Car Following," *Operations Research*, **6**(2), 165–184, 1958.

R. E. Chandler, E. W. Montroll, R. B. Potts, and R. W. Rothery, "Traffic Dynamics: Analysis of Stability in Car Following," *Operations Research*, **7**(1), 86–106, 1959.

B. D. Greenshields, "A Study in Highway Capacity," *Highway Research Board Proceedings*, **14**, 1935.

R. Haberman, *Mathematical Models*, Prentice-Hall, Englewood Cliffs, NJ, 1977.

G. Newell, "A Simplified Theory of Kinematic Waves: I General Theory; II Queuing at Freeway Bottlenecks; III Multi-destination Flows," *Transportation Research-B*, **27B**, 281–313, 1991.

W. W. Recker, "Understanding the Nature of Traffic," *Notes for CE122–Transportation Systems II: Operations and Control*, University of California, Irvine, CA, Winter 2003.

J. A. Wattleworth, "Traffic Flow Theory," in J. E. Baerwald (Ed.), *Transportation and Traffic Engineering Handbook*, Prentice-Hall, Englewood Cliffs, NJ, 1976.

M. Wohl and B. V. Martin, *Traffic Systems Analysis*, McGraw-Hill, New York, 1967.

6.6 Problems

6.16. What is the meaning and physical significance of the statement, $\partial q/\partial x > 0$, (i.e., that the macroscopic traffic flow rate, $q(x,t)$, increases with the distance, x, along the line of traffic)?

6.17. If the average length of a car (in pre-*Expedition* days) is 5 m, what is the density of traffic in a line when its cars are maintaining a distance of two car lengths between themselves. What is the traffic flow if the line is moving at 80 km/hr (50 mph)? (*Hint*: You may ignore the fact that the data given ignores both AAA recommendations and your own experience on a freeway or turnpike.)

6.18. (a) Assume that velocity depends linearly on density, such that $v(\rho) = a + b\rho$. Determine the values of a and b in terms of the maximum values of the speed and the density, assuming that the assumptions of eqs. (6.11a–c) hold.

(b) How does the flow depend on the density?

6.19. (a) Sketch the fundamental diagram of road traffic for the model developed in Problem 6.18 if $a = 80$ km/hr and $b = -10^5\ \text{m}^2/\text{car}\cdot\text{hr}$.

(b) Determine the values of the density and the speed when the flow is a maximum.

(c) What is the capacity of the road being modeled?

6.20. Consider a flow-density relationship of the form $q(\rho) = \rho(\alpha - \beta\rho)$. The best fit (i.e., least squares) of this relationship to some real traffic data occurred when $\alpha = 91.33$ km/hr and $\beta = 1.4\ \text{km}^2/\text{car}\cdot\text{hr}$.

(a) What is the maximum density?

(b) What is the maximum speed?

(c) What is the capacity of the road?
(d) Identify the type of road being modeled and explain your identification.

6.21. Find the speed of traffic on a line of traffic for which there are three car lengths between the leader and follower cars. (*Hint*: Use macroscopic traffic theory with a linear speed-density relation.)

6.22. Determine the capacity of the road described in Problem 6.21 if cars are assumed to be 5 m long, $v_{\text{max}} = 88.5$ km/hr and $\rho_{\text{max}} = 0.22^{-1}$.

6.23. The data in the table shown below were obtained by recording the indicated parameters along a busy stretch of highway.

(a) Sketch the fundamental diagram for this traffic flow.
(b) What is the maximum traffic flow?
(c) What are the density and speed at the maximum flow rate?

Speed (mph)	Density (cars/mi)
42	44
40	49
37	53
35	58
32	64
28	67
26	69
23	74
20	80
19	85
18	90
17	95
16	101
15	106
14	112
13	120
12	128
11	139
10	151
9	166

6.24. Plot traffic speed against traffic density for the data given in Problem 6.23. Draw an approximate curve through this data and estimate the maximum values of the speed and the density on this road.